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Introductory Linear Operator Theory

In this chapter we apply concepts of functional analysis, especially those concepts related to Hilbert and Banach spaces, to introduce basic operator theory relevant to applied electromagnetics. We begin with the definition of a linear operator and provide examples of common operators that arise in physical problems. We next define linear functionals as a special class of linear operators. Linear functionals occur quite often in electromagnetics and are very useful in theoretical investigations and in formulating problems to be solved numerically. In addition, the concept of a linear functional, in conjunction with the Riesz representation theorem, gives an appropriate motivation for introducing the important concept of an adjoint operator. Next, the class of self-adjoint operators is discussed, as well as the broader category of normal operators. We will see later that self-adjoint operators, and especially compact self-adjoint operators, have very nice mathematical properties that can be usefully exploited. Definite operators are then discussed, which themselves are contained within the class of self-adjoint or symmetric operators, and lead to a new notion of convergence. Compact operators are introduced, both at the function and sequence (infinite matrix) levels, and examples from applied mathematics and electromagnetics are provided. The important topic of the operator inverse is then covered, which naturally arises in the consideration of solving operator equations and, primarily for differential operators, is related to the concept of a Green's function. Projection operators are briefly discussed, followed by a statement of the Fredholm alternative, which details solvability conditions for certain operator equations.

3.1 Linear Operators

3.1.1 General Concepts

We will define a *linear operator* as a linear mapping between linear spaces.

Definition 3.1. A mapping A from X to Y , denoted as $A : X \rightarrow Y$, called a *linear operator* (linear mapping, linear transformation) if for all x, y in the domain of A (defined below) and $\alpha, \beta \in \mathbf{C}$,

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay,$$

where $Ax, Ay \in Y$.

Note that a linear operator maps elements of X to elements of Y , and often X is equal to Y . As an example of a familiar operator, the mapping $A : \mathbf{C}^n \rightarrow \mathbf{C}^m$, given by an $m \times n$ complex-valued matrix mapping n -component vectors in $\mathbf{C}^n$ into m -component vectors in $\mathbf{C}^m$.

The elements on which the operator acts, and the elements it produces are given special names.

Definition 3.2. The domain of a linear operator $A : X \rightarrow Y$, denoted as D_A , is simply the set of elements for which the mapping A is defined.

It is convenient in the following to restrict our linear spaces X and Y to be Hilbert spaces, denoted as H_1 and H_2 , respectively, although in a few instances we will consider Banach-space formulations.

For operators acting on a Hilbert space H_1 , $D_A = H_1$. Often we will be interested in operators acting in a Hilbert space H_1 , specifically on a closed linear subspace M of a Hilbert space. In this case $D_A = M \subset H_1$, $D_A \neq H_1$. For example, the differential operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by $(Ax)(t) = dx(t)/dt = x'(t)$ cannot have as its domain all of $\mathbf{L}^2(a, b)$, since many functions in this space are not differentiable, or even continuous.

Defining a domain is an important part of defining an operator; an operator is completely defined by specifying both its domain and its action on elements of the domain. Furthermore, often properties of an operator are very sensitive to the choice of the domain. This is especially true for differential operators, where boundary conditions are part of the domain specification.

The most important operators acting in Hilbert spaces are *densely defined*.

Definition 3.3. An operator $A : H_1 \rightarrow H_2$ acting in H_1 is said to be *densely defined* if D_A is an everywhere dense subset of H_1 , i.e., $\tilde{D}_A = H_1$.

Recall from Definition 2.13 that this means that every element in H_1 can be approximated arbitrarily closely by elements of the set D_A , and any neighborhood of an arbitrary point of H_1 contains points from the set D_A . Thus, in this case while D_A is not the whole space H_1 (H_1 may, in fact, contain infinitely many elements not in D_A), D_A is “close” to H_1 in this sense.

Unless otherwise noted, when $D_A \neq H_1$ we will work with operators on densely defined subspaces of H_1 .

Definition 3.4. *The range (or image space) of a linear operator $A : H_1 \rightarrow H_2$, denoted as R_A , is the set of elements $y \in H_2$ resulting from Ax for all possible $x \in D_A$.*

The dimension of the range of A is called the *rank* of A , and for $Ax = y$, y is called the *image* of x . Note that the range of A need not be all of H_2 . A mapping $A : H_1 \rightarrow H_2$ for which $R_A = H_2$ is said to be *onto* (surjective), whereas if $R_A \subset H_2$ is a proper subspace of H_2 then the mapping is *into*. Therefore, $A : H_1 \rightarrow H_2$ may be *in* or *on* H_1 , and *into* or *onto* H_2 .

Definition 3.5. *The null space (or kernel) of a linear operator $A : H_1 \rightarrow H_2$, denoted as N_A , is the set of all elements $x \in D_A$ for which $Ax = 0$.*

As an example, the common vector identity $\nabla \times \nabla \psi = \mathbf{0}$ indicates that the range of the gradient operator must be in the null space of the curl operator, i.e., $R_\nabla \subseteq N_{\nabla \times}$. The dimension of the null space is called the *nullity*. It is easy to see that the range and null space of an operator are linear subspaces of Hilbert spaces.

An important class of operators are those having *finite rank*.¹ This is a different concept from that of an operator acting on a finite-dimensional domain. It can be shown that for $A : H_1 \rightarrow H_2$ a linear operator and H_1 finite-dimensional, then R_A is a finite-dimensional subspace of H_2 (and $\dim(H_1) = \text{nullity}(A) + \text{rank}(A)$). It then follows that every operator acting on a finite-dimensional domain is a finite-rank operator, yet the converse is not necessarily true. In the following we reserve the name “finite-dimensional operator” for the special subset of finite-rank operators that act on finite-dimensional spaces.

Boundedness and Continuity of Operators

A very important classification of operators pertains to their *boundedness*.

Definition 3.6. *A linear operator $A : H_1 \rightarrow H_2$ is bounded on its domain if, for all $x \in H_1$, there exists a number $k > 0$ such that²*

$$\|Ax\|_2 \leq k \|x\|_1.$$

¹ Also known as *finite-dimensional operators*.

² As always when working in Hilbert spaces, we exclusively use the norm induced by the inner product, with the subscript (often omitted) indicating the associated space (H_1 or H_2).

We use $x \in H_1$ rather than $x \in D_A$ since it is shown in Section 3.4 that for a bounded operator A , $D_A = H_1$, at least by extension.

Note that bounded operators map bounded sets in D_A into bounded sets in R_A . This is a somewhat different notion than the calculus idea of a bounded function, where a function is said to be bounded if the range of the function is a bounded set.

If an operator is not bounded, it is said to be *unbounded*. To show that an operator A is unbounded it is sufficient to find a sequence of elements $\mathbf{x} = \{x_1, x_2, \dots\}$ with $x_n \in D_A$ such that $\|x_n\|_1 \leq M$ for some M and $\|Ax_n\|_2 \rightarrow \infty$ as $n \rightarrow \infty$ (see, e.g., Example 8 in this section). The most important class of unbounded operators relevant to electromagnetics is differential operators.

The definition of boundedness of an operator brings about the concept of an operator norm (as an abstraction of size).

Definition 3.7. The norm of an operator $A : H_1 \rightarrow H_2$, denoted as $\|A\|$, is the smallest number k that satisfies $\|Ax\|_2 \leq k \|x\|_1$ for all $x \in H_1$. This can be stated as

$$\|A\| = \sup_{\|x\|_1 \neq 0} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1 \leq 1} \|Ax\|_2 = \sup_{\|x\|_1 = 1} \|Ax\|_2.$$

It is convenient to note that $\|Ax\|_2 \leq \|A\| \|x\|_1$. In the following we will generally omit the subscript on the norm.

A fundamental property that divides operators into two classes is continuity, or lack thereof.

Definition 3.8. A linear operator $A : H_1 \rightarrow H_2$ is said to be continuous at $x_0 \in H_1$ if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that $\|Ax - Ax_0\| < \epsilon$ whenever $\|x - x_0\| < \delta$.

The next theorem states an extremely important fact concerning continuous operators.

Theorem 3.1. If $A : H_1 \rightarrow H_2$ is a continuous linear operator at $x_0 \in H_1$, and if $x_n \rightarrow x_0$, then $Ax_n \rightarrow Ax_0$.

This theorem states that if $\lim_{n \rightarrow \infty} x_n = x_0$, then $\lim_{n \rightarrow \infty} Ax_n = A \lim_{n \rightarrow \infty} x_n$, indicating that the operator and the limit operation may be interchanged.

Proof. Assume $x_n \rightarrow x$ (i.e., $\|x_n - x\| \rightarrow 0$) and show $\|Ax_n - Ax\| \rightarrow 0$ (i.e., $Ax_n \rightarrow Ax$). Indeed, $\|Ax_n - Ax\| = \|A(x_n - x)\| \leq \|A\| \|x_n - x\|$. Because $\|x_n - x\| \rightarrow 0$, then $\|Ax_n - Ax\| \rightarrow 0$, where we assume $\|A\| < \infty$. ■

A very important relationship exists between the continuity and boundedness of a linear operator.

Theorem 3.2. *A linear operator $A : H_1 \rightarrow H_2$ is continuous if and only if it is bounded.*

Proof. Assume A is bounded and x_0 an arbitrary point in H_1 . Then $\|Ax - Ax_0\| = \|A(x - x_0)\| \leq k\|x - x_0\|$. Thus if $\|x - x_0\| < \delta$, setting $\delta = \varepsilon/k$ gives $\|A(x - x_0)\| < \varepsilon$. Then A is continuous at an arbitrary point and so is continuous on its domain.

Now assume A is continuous, and in particular take $x_0 = 0$. Then for $\varepsilon = 1$ there exists a $\delta > 0$ such that $\|Ax\| < 1$ whenever $\|x\| < \delta$. For $x \neq 0$ let $z = \beta x$ with $\beta = \delta/\|x\|$. Then $\|z\| = \|\beta x\| = \|(\delta x)/\|x\|\| = \delta$. Therefore, $1 > \|Az\| = \|A(\beta x)\| = |\beta| \|Ax\|$ and $\|Ax\| < 1/\beta = (1/\delta) \|x\|$ and so A is bounded. Of course, for $x = 0$, $\|A0\| = 0 < \|x\|$. ■

In summary, continuity at 0 implies boundedness, which implies continuity at an arbitrary point x .

Theorem 3.3. *If a linear operator $A : H_1 \rightarrow H_2$ is continuous at one point, it is continuous on its domain.*

If a linear operator is continuous, it is uniformly continuous, since δ in the above proof is independent of x_0 .

For linear operators (but not most ordinary functions) boundedness implies continuity, and conversely. Unbounded operators are then related to some sort of discontinuity of the operator. For unbounded operators, $x_n \rightarrow x_0$ does not imply $Ax_n \rightarrow Ax_0$.

It can be shown that the space of all bounded operators $A : X_1 \rightarrow X_2$, where $X_{1,2}$ are Banach or Hilbert spaces, is itself a Banach space [1, p. 43]. This is important since later we need the concept of convergence of a series of operators (see, e.g., Section 4.4), and in a complete space absolute convergence, $\sum_n \|A^n\|$, implies convergence of $\sum_n A^n$.

Theorem 3.4. *If $A : H_1 \rightarrow H_2$, and $B : H_2 \rightarrow H_3$ are bounded linear operators, then the composition $BA : H_1 \rightarrow H_3$ is bounded, and $\|BA\| \leq \|B\| \|A\|$.*

Proof. $\|BAx\| \leq \|B\| \|Ax\| \leq \|B\| \|A\| \|x\|$, proving boundedness of BA . The inequality follows from $\|BAx\| \leq \|B\| \|Ax\|$ and the definition of $\|BA\|$. ■

Finally, there is an interesting statement about continuity of operators on finite-dimensional spaces.

Theorem 3.5. *Consider a linear operator $A : H_1 \rightarrow H_2$. If H_1 is finite-dimensional, then A is continuous (and so bounded).*

Proof. Let $\{x_n\}_{n=1}^N$ be a basis for H_1 , and take any $0 \neq x \in H_1$ as $x = \sum_{n=1}^N \alpha_n x_n$. Then $\|Ax\| = \left\| \sum_{n=1}^N \alpha_n Ax_n \right\| \leq \sum_{n=1}^N |\alpha_n| \|Ax_n\|$. Let

$K = \max_n \{\|Ax_n\|\}$. Then $\|Ax\| \leq K \sum_{n=1}^N |\alpha_n|$. It can be shown (proof is omitted here; it is at least reasonable since the sum is finite) that an M exists such that $\sum_{n=1}^N |\alpha_n| \leq M \|x\|$ and so $\|Ax\| \leq KM \|x\|$, which shows that A is bounded and hence continuous. ■

3.1.2 Examples Related to Linear Operators

1. Ordinary real-valued functions $x(t)$ can be thought of as operators $x : \mathbf{R} \rightarrow \mathbf{R}$. However, only linear functions are themselves linear operators. For example, the function $x(t) = t$ for $t \in \mathbf{R}$ defines a linear operator, yet $x(t) = \sin(t)$ (or $t + 1, t^2, e^t$, etc.) does not.
2. The common “real-part” and “imaginary-part” operators $A_r, A_i : \mathbf{C} \rightarrow \mathbf{R}$ defined by

$$\begin{aligned} A_r x &\equiv \operatorname{Re}(x), \\ A_i x &\equiv \operatorname{Im}(x), \\ D_{A_r} &= D_{A_i} \equiv \mathbf{C} \end{aligned}$$

are easily seen to be nonlinear operators. The complex conjugation operator $A_{cc} : \mathbf{C} \rightarrow \mathbf{C}$ defined by

$$\begin{aligned} A_{cc} x &\equiv \bar{x}, \\ D_{A_{cc}} &\equiv \mathbf{C}, \end{aligned}$$

is also nonlinear.

3. The identity operator $I : H \rightarrow H$ defined by

$$\begin{aligned} Ix &\equiv x, \\ D_I &\equiv H, \end{aligned}$$

is a bounded linear operator with $\|I\| = 1$, $R_I = H$, and $N_I = \{0\}$.

4. The null operator $0 : H \rightarrow H$ defined by

$$\begin{aligned} 0x &\equiv 0, \\ D_0 &\equiv H, \end{aligned}$$

is a bounded linear operator with $\|0\| = 0$, $R_0 = \{0\}$, and $N_0 = H$.

5. A bounded linear operator $A : H \rightarrow H$ defined by

$$\begin{aligned} Ax &= y, \\ D_A &\equiv H, \end{aligned}$$

where H is an infinite-dimensional space, is uniquely represented by an infinite matrix with respect to a given basis. To see this, let $\{x_1, x_2, \dots\}$ be an orthonormal basis for H . Then, for any $x \in H$, $x = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i$ and by continuity³ and linearity, $Ax = \sum_{i=1}^{\infty} \langle x, x_i \rangle Ax_i = y$. Taking the inner product of both sides with x_j , $j = 1, 2, \dots$, leads to the system of equations

$$\sum_{i=1}^{\infty} \langle x, x_i \rangle \langle Ax_i, x_j \rangle = \langle y, x_j \rangle, \quad j = 1, 2, \dots,$$

which can be written in matrix form as

$$\begin{bmatrix} \langle Ax_1, x_1 \rangle & \langle Ax_2, x_1 \rangle & \cdots \\ \langle Ax_1, x_2 \rangle & \langle Ax_2, x_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \langle x, x_1 \rangle \\ \langle x, x_2 \rangle \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle y, x_1 \rangle \\ \langle y, x_2 \rangle \\ \vdots \end{bmatrix}.$$

Therefore, the bounded linear operator $A : H \rightarrow H$ defined by $Ax = y$ is represented in the basis $\{x_1, x_2, \dots\}$ by the infinite matrix $[a_{ij}]$, where $a_{ij} = \langle Ax_j, x_i \rangle$. It can be shown that $\sum_{i=1}^{\infty} |a_{ij}|^2 < \infty$ and $\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$. Conversely, an infinite matrix $[a_{ij}]$, where $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$ defines a bounded⁴ linear operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$, with respect to a certain basis, with

$$\|A\|^2 \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2.$$

The number $\left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2\right)^{1/2}$ is called the *absolute norm* of the operator A .

Similarly, a bounded operator $A : H_1 \rightarrow H_2$ defined by $Ax = y$, $D_A \equiv H_1$, where H_1, H_2 are infinite-dimensional spaces, is uniquely represented with respect to given bases by an infinite matrix $[a_{ij}]$, with $a_{ij} = \langle Ax_j, y_i \rangle$ such that

$$\begin{bmatrix} \langle Ax_1, y_1 \rangle & \langle Ax_2, y_1 \rangle & \cdots \\ \langle Ax_1, y_2 \rangle & \langle Ax_2, y_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \langle x, x_1 \rangle \\ \langle x, x_2 \rangle \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle y, y_1 \rangle \\ \langle y, y_2 \rangle \\ \vdots \end{bmatrix},$$

³While any linear operator can be interchanged with a finite sum, boundedness (continuity) of A is required to allow the free interchange $A \sum_{i=1}^{\infty} \alpha_i x_i = \sum_{i=1}^{\infty} \alpha_i Ax_i$.

⁴The conditions $\sum_{i=1}^{\infty} |a_{ij}|^2 < \infty$ and $\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$ are not sufficient for A to be bounded; the condition $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$ is a sufficient but not necessary condition for A to be bounded. It can also be shown that the conditions $\sum_{i=1}^{\infty} |a_{ij}| < K$ and $\sum_{j=1}^{\infty} |a_{ij}| < K$, where K does not depend on i or j , are also sufficient conditions for A to be bounded.

where $\{x_1, x_2, \dots\}$ is an orthonormal basis for H_1 , and $\{y_1, y_2, \dots\}$ is an orthonormal basis for H_2 . If $\langle x_i, y_j \rangle = \delta_{ij}$, the two bases are called *bi-orthonormal*.

6. As a finite-dimensional version of a matrix operator, the operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^m$ (bounded since $\mathbb{C}^n$ is finite-dimensional) is uniquely represented, for $\{x_1, x_2, \dots, x_n\}$ a basis of $\mathbb{C}^n$ and $\{y_1, y_2, \dots, y_m\}$ a basis of $\mathbb{C}^m$, by the $m \times n$ matrix $[a_{ij}]$ where $a_{ij} = \langle Ax_j, y_i \rangle$. Conversely, every $m \times n$ matrix $[a_{ij}]$ where $a_{ij} = \langle Ax_j, y_i \rangle$ uniquely defines a bounded linear operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^m$ with respect to the given bases. Exactly the same result is obtained for a more general mapping $A : H_1 \rightarrow H_2$, where H_1, H_2 are any two finite-dimensional Hilbert spaces.

For operators on finite-dimensional spaces some explicit equations for the norm may be developed in terms of the matrix representation of the operator. For instance, considering the operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^m$ and using the norm $\|\mathbf{x}\|_1 = \sum_{j=1}^n |x_j|$ lead to the matrix norm⁵

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|, \quad (3.1)$$

which obviously corresponds to summing the absolute values of elements of each column and taking the largest such sum. This is called the *column-sum norm*. Alternatively, the ∞ -norm $\|\mathbf{x}\|_\infty = \max_{1 \leq j \leq n} |x_j|$ leads to

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|,$$

which corresponds to summing the absolute values of elements of each row and taking the largest such sum (*row-sum norm*). Obviously the relationship $\|A\|_1 = \|A^T\|_\infty$ holds. The two-norm $\|\mathbf{x}\|_2 = (\sum_{j=1}^n |x_j|^2)^{1/2}$ leads to $\|A\|_2 = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue of A^*A , which is sometimes called the *spectral norm* of A .

The above are known as *subordinate matrix norms*, since they arise directly from an associated vector norm. Other matrix norms, which come directly from Definition 2.31, rather than the operator norm Definition 3.7, are sometimes found to be useful. Two common

⁵To see this, consider

$$\|A\mathbf{x}\| = \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij} x_j \right| \leq \sum_{j=1}^n \left(\sum_{i=1}^m |a_{ij}| \right) |x_j| \leq \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| \|\mathbf{x}\|.$$

The maximum over all $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ leads to the desired formula.

norms from this latter class are the *Schur norm*, defined as

$$\|A\|_S = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2},$$

and the *maximum norm* $\|A\|_M = \max_{i,j} |a_{ij}|$. The Schur norm satisfies $\|AB\| \leq \|A\| \|B\|$, while the maximum norm does not. Both are easy to compute but are generally less useful than the subordinate norms. Some discussion of these concepts, as well as the role of the *condition number*⁶ in numerical matrix analysis, can be found in [2, pp. 176–183].

7. Consider the operators $A, B : \mathbf{C}^n \rightarrow \mathbf{C}^n$ associated with matrices $[a_{ij}]_{i,j=1}^n$ and $[b_{ij}]_{i,j=1}^n$, respectively. The composition (product) $C = AB$ of operators A and B is associated with the product of corresponding matrices:

$$(C\mathbf{x})_{i=1}^n = (AB\mathbf{x})_{i=1}^n = \sum_{j=1}^n \left(\sum_{k=1}^n a_{ik} b_{kj} \right) x_j = \sum_{j=1}^n c_{ij} x_j,$$

where $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$. If A and B are bounded in $\mathbf{C}^n$, then by Theorem 3.4 the product C is bounded in $\mathbf{C}^n$. Indeed,

$$\begin{aligned} \|C\mathbf{x}\| &= \sum_{i=1}^n \left| \sum_{j=1}^n \left(\sum_{k=1}^n a_{ik} b_{kj} \right) x_j \right| \\ &\leq \max_{1 \leq k \leq n} \sum_{i=1}^n |a_{ik}| \max_{1 \leq j \leq n} \sum_{k=1}^n |b_{kj}| \|\mathbf{x}\|. \end{aligned}$$

Similarly, the product $C = AB : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ of bounded in $\mathbf{l}^2$ operators A and B corresponding to the infinite matrices $[a_{ik}]$ and $[b_{kj}]$, assuming $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|^2 < \infty$ and $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |b_{kj}|^2 < \infty$, is bounded in $\mathbf{l}^2$. This property of bounded operators can be used to simplify the proof of boundedness and compactness of the product operator obtained in some electromagnetic problems, where each matrix element is obtained in terms of infinite series (Galerkin-type projection techniques often result in such internal summations).

8. The differential operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$\begin{aligned} (Ax)(t) &\equiv \frac{dx(t)}{dt} = x'(t), \\ D_A &\equiv \{x : x, x' \in \mathbf{L}^2(a, b)\} \end{aligned}$$

⁶The condition number of a matrix A is defined as $\|A\| \|A^{-1}\|$ using a subordinate matrix norm.

is unbounded. To see this, consider the sequence of functions $x_n(t) = \cos nt$, $n = 1, 2, \dots$, defined on the interval $[-\pi, \pi]$. Then

$$\|x_n\| = \left(\int_{-\pi}^{\pi} (\cos nt)^2 dt \right)^{1/2} = \sqrt{\pi}$$

and

$$\|Ax_n\| = \left(\int_{-\pi}^{\pi} (n \sin nt)^2 dt \right)^{1/2} = n\sqrt{\pi},$$

resulting in $\|Ax_n\| = n\|x_n\|$. Clearly, $\|Ax_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, all differential operators are usually unbounded, although distributional spaces can be constructed where they are, in fact, bounded.

As mentioned previously, this operator cannot have as its domain all of $\mathbf{L}^2(a, b)$ since many functions in this space are not differentiable, or even continuous. Note that N_A is the set of constant functions.

9. The multiplication operator $A : \mathbf{C}(\Omega) \rightarrow \mathbf{C}(\Omega)$ ($\mathbf{C}(\Omega)$ is a Banach space in the max-norm, $\Omega \subset \mathbf{R}^n$) defined by

$$(Ax)(\mathbf{t}) \equiv f(\mathbf{t})x(\mathbf{t})$$

is a bounded linear operator assuming $k = \max_{\mathbf{t} \in \Omega} |f(\mathbf{t})| < \infty$ with $\|A\| \leq k$. Boundedness follows from the inequality

$$|f(\mathbf{t})x(\mathbf{t})| \leq \|f\| \|x\|,$$

where $\|f\| = \max_{\mathbf{t} \in \Omega} |f(\mathbf{t})|$ and $\|x\| = \max_{\mathbf{t} \in \Omega} |x(\mathbf{t})|$. Also, the multiplication operator A is bounded in the Hilbert space $\mathbf{L}^2(\Omega)$ for f continuous,

$$\|Ax\|^2 = \int_{\Omega} |f(\mathbf{t})x(\mathbf{t})|^2 d\Omega \leq k^2 \|x\|^2,$$

where $k = \max_{\mathbf{t} \in \Omega} |f(\mathbf{t})| < \infty$.

10. The Banach-space integral operator $A : \mathbf{C}(\Omega) \rightarrow \mathbf{C}(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} k(\mathbf{t}, \mathbf{s})x(\mathbf{s}) d\Omega,$$

where $k(\mathbf{t}, \mathbf{s}) \in \mathbf{C}(\Omega \times \Omega)$ and $\Omega \subset \mathbf{R}^n$, with the max-norm

$$\|k\| = \max_{\mathbf{t} \in \Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\Omega < \infty,$$

is bounded with

$$\|A\| \leq \max_{\mathbf{t} \in \Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\Omega.$$

Indeed, using the inequality of the previous example,

$$|(Ax)(\mathbf{t})| \leq \int_{\Omega} |k(\mathbf{t}, \mathbf{s})| |x(\mathbf{s})| d\Omega \leq \max_{\mathbf{t} \in \Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\Omega \|x\|$$

and, in fact, it can be shown the equality holds.

Similarly, it can be shown that the weakly singular integral operator

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} \frac{k(\mathbf{t}, \mathbf{s})}{|\mathbf{t} - \mathbf{s}|^{\alpha}} x(\mathbf{s}) d\Omega$$

is bounded in $\mathbf{C}(\Omega)$ for $\alpha < n$ (for $\alpha = 0$ this reduces to the previous case).

11. The Hilbert-space integral operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} k(\mathbf{t}, \mathbf{s}) x(\mathbf{s}) d\Omega$$

with kernel $k(\mathbf{t}, \mathbf{s}) \in \mathbf{L}^2(\Omega \times \Omega)$ and the two-norm

$$\|k\|^2 = \int_{\Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})|^2 d\Omega_s d\Omega_t < \infty,$$

where $\Omega \subset \mathbf{R}^n$, is bounded with⁷

$$\|A\|^2 \leq \int_{\Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})|^2 d\Omega_s d\Omega_t.$$

The above follows from the inequality

$$\begin{aligned} \|Ax\|^2 &= \int_{\Omega} |(Ax)(\mathbf{t})|^2 d\Omega_t = \int_{\Omega} \left| \int_{\Omega} k(\mathbf{t}, \mathbf{s}) x(\mathbf{s}) d\Omega_s \right|^2 d\Omega_t \\ &\leq \int_{\Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})|^2 d\Omega_s d\Omega_t \|x\|^2. \end{aligned}$$

Such operators are known as *Hilbert-Schmidt operators*, and the kernel $k(\mathbf{t}, \mathbf{s})$ is said to be of *Hilbert-Schmidt type*.

More generally, a bounded operator $A : H \rightarrow H$ is called a Hilbert-Schmidt operator if, given $\{x_n\}$ an orthonormal basis of H ,

$$\sum_n \|Ax_n\|^2 < \infty.$$

⁷A sufficient condition of boundedness of the integral operator in $\mathbf{L}^2(\Omega)$ is $\int_{\Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})|^2 d\Omega_s d\Omega_t < \infty$. Also, the integral operator is bounded in $\mathbf{L}^2(\Omega)$ if there is a number $M < \infty$ such that $\int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\Omega_s \leq M$ and $\int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\Omega_t \leq M$. This condition of boundedness is also sufficient.

12. The integration (*Volterra*) operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv \int_a^t x(s) ds$$

is a linear bounded operator for $-\infty < a < b < \infty$. This is just a special case of the last example ($\Omega = (a, b)$) with $k(t, s) = H(t - s)$, the Heaviside function, where $H(t - s) = 1$ for $t > s$, and 0 for $t < s$.

13. Let $k(t, s)$ be bounded, i.e., $k(t, s) \in \mathbf{L}^\infty((a, b) \times (a, b))$. The weakly singular integral operator

$$(Ax)(t) \equiv \int_a^b \frac{k(t, s)}{|t - s|^\alpha} x(s) ds$$

is bounded in $\mathbf{L}^p(a, b)$ for $1 < p < \infty$ and $0 \leq \alpha < 1$ [3, p. 57]. Moreover, with $\alpha = 1$ the operator, taken in the sense of the Cauchy principal value, is bounded in $\mathbf{L}^p(a, b)$ for $1 < p < \infty$. As a special case, the Hilbert transform operator defined as

$$(Ax)(t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(s)}{(t - s)} ds$$

is bounded in $\mathbf{L}^p(-\infty, \infty)$ for $1 < p < \infty$.

Generalizing, the weakly singular integral operator defined by

$$(Af)(\mathbf{s}) \equiv \int_{\Omega} \frac{k(\mathbf{t}, \mathbf{s})}{|\mathbf{t} - \mathbf{s}|^\alpha} x(\mathbf{s}) d\Omega$$

with $0 \leq \alpha < n$ and where $k(\mathbf{t}, \mathbf{s})$ is bounded on $\Omega \times \Omega$, is bounded in $\mathbf{L}^2(\Omega)$ for $\Omega \subset \mathbf{R}^n$ [4, p. 162]. This also holds for integrals of the form

$$(Af)(\mathbf{s}) \equiv \int_{\Gamma} \frac{k(\mathbf{t}, \mathbf{s})}{|\mathbf{t} - \mathbf{s}|^\alpha} x(\mathbf{s}) d\Gamma,$$

where Γ is a smooth n -dimensional surface in an $(n + 1)$ -dimensional space. Therefore, in a finite region of three-dimensional space, the usual free-space Green's function kernel

$$k(\mathbf{r}, \mathbf{r}') = (e^{-ik|\mathbf{r} - \mathbf{r}'|}) / (4\pi|\mathbf{r} - \mathbf{r}'|)$$

generates a bounded operator in $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Gamma)$.

14. The singular Cauchy integral operator defined by

$$(Ax)(t) \equiv \frac{1}{\pi i} \int_{\Gamma} \frac{x(s)}{s - t} ds$$

is bounded on the space $\mathbf{H}_{0,\alpha}(\Gamma)$ for every α such that $0 < \alpha < 1$, where Γ is a smooth line [3], [5].

15. Consider the integral operators A_1 and A_2 acting in the Banach space $\mathbf{C}(a, b)$ with kernels $k_1(t, \tau)$ and $k_2(\tau, s)$, respectively [6]. The integral product operator $A = A_1 A_2$ for any function $x \in \mathbf{C}(a, b)$ is defined as

$$\begin{aligned}
 (Ax)(t) &= (A_1 A_2 x)(t) = \int_a^b k_1(t, \tau) (A_2 x)(\tau) d\tau \\
 &= \int_a^b k_1(t, \tau) \left(\int_a^b k_2(\tau, s) x(s) ds \right) d\tau \\
 &= \int_a^b \left(\int_a^b k_1(t, \tau) k_2(\tau, s) d\tau \right) x(s) ds \\
 &= \int_a^b k^{(2)}(t, s) x(s) ds.
 \end{aligned}$$

Thus, $A = A_1 A_2$ is an integral operator with the kernel $k^{(2)}(t, s) = \int_a^b k_1(t, \tau) k_2(\tau, s) d\tau$. If the integral operators A_1 and A_2 are bounded in $\mathbf{C}(a, b)$ (i.e., $\|A_1\|, \|A_2\| < \infty$), then the integral product operator $A = A_1 A_2$ is bounded in $\mathbf{C}(a, b)$ (this simply follows from $\|Ax\| \leq \|A_1\| \|A_2\| \|x\|$). Also, it can be shown if the integral operators A_1 and A_2 are bounded in the space $\mathbf{L}^2(a, b)$, then the integral product operator $A = A_1 A_2$ is bounded in $\mathbf{L}^2(a, b)$.

The integral product operator has been utilized in the method of overlapping regions (see Section 9.5) and in the Schwarz's iterative method where the resulting kernel is determined by subsequent iterations. Indeed, if we set $k_1 = k_2 = k$ such that $k^{(2)}(t, s) = \int_a^b k(t, \tau) k(\tau, s) d\tau$, then

$$k^{(n)}(t, s) = \int_a^b k(t, \tau) k^{(n-1)}(\tau, s) d\tau, \quad n > 2.$$

3.1.3 Isomorphisms

Since we have the concept that an operator takes elements of one space to those of another space, it is worthwhile to consider some relationships between those spaces. The following definitions are useful.

- Two linear spaces X and Y are said to be *isomorphic* if there is a one-to-one, onto linear map $A : X \rightarrow Y$.
- Two normed spaces X and Y are said to be (topologically) *isomorphic* if there is a continuous one-to-one, onto linear map $A : X \rightarrow Y$ such that A^{-1} is also continuous. Furthermore, the spaces are said to be *isometrically isomorphic* or *unitarily equivalent* if A is unitary, i.e.,

A preserves norms ($\|Ax\| = \|x\|$ for all $x \in X$). It can be shown that two Hilbert spaces are isomorphic if and only if they are isometrically isomorphic.

An operator $A : X \rightarrow Y$ that connects two isomorphic spaces is called an *isomorphism*. If two spaces are isomorphic to one another, then they are essentially the same space, at least at an abstract level.

As a simple example, the normed spaces $\mathbf{C}$ and $\mathbf{R}^2$ are easily seen to be isometrically isomorphic, with $A : \mathbf{C} \rightarrow \mathbf{R}^2$ the mapping that takes $z = x + iy \in \mathbf{C}$ into the ordered pair $(\operatorname{Re} z, \operatorname{Im} z) = (x, y) \in \mathbf{R}^2$. Indeed, $\|Az\| = \sqrt{x^2 + y^2} = |z| = \|z\|$.

One reason for introducing the concept of spaces being isomorphic to one another is to see the connection between function spaces and sequence spaces. For example, consider the space $\mathbf{L}^2(-\pi, \pi)$. Every $f \in \mathbf{L}^2(-\pi, \pi)$ can be written as a Fourier series

$$f = \sum_{n=-\infty}^{\infty} a_n \frac{e^{inx}}{\sqrt{2\pi}},$$

where $a_n = \langle f, e^{inx}/\sqrt{2\pi} \rangle = (1/\sqrt{2\pi}) \int_{-\pi}^{\pi} f(x) e^{-inx} dx$, and it can be easily shown that $\{a_n\} \in \mathbf{l}^2$. Let the operator $A : \mathbf{L}^2(-\pi, \pi) \rightarrow \mathbf{l}^2$ be the mapping that takes $f \in \mathbf{L}^2(-\pi, \pi)$ to the sequence $\{a_n\}_{n=-\infty}^{\infty}$ as

$$(Af)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \{a_n\}$$

with the inverse mapping $A^{-1} : \mathbf{l}^2 \rightarrow \mathbf{L}^2(-\pi, \pi)$ taking $\{a_n\}$ to f as

$$A^{-1}(\{a_n\}) = \sum_{n=-\infty}^{\infty} a_n \frac{e^{inx}}{\sqrt{2\pi}} = f(x).$$

It is clear that A is linear, one-to-one (if $a_n = \langle f, e^{inx}/\sqrt{2\pi} \rangle = 0$ for all n , then $f = 0$ since $\{e^{inx}/\sqrt{2\pi}\}$ is complete in $\mathbf{L}^2(-\pi, \pi)$), and onto (every element of $\mathbf{l}^2$ corresponds to an infinite series, representing a function in $\mathbf{L}^2$; see Section 2.5.7). Therefore, $\mathbf{L}^2(-\pi, \pi)$ and $\mathbf{l}^2(-\infty, \infty)$ are isomorphic with the Fourier series operator providing the required isomorphism. Since both spaces are Hilbert spaces, they are isometrically isomorphic.⁸ Furthermore, any separable Hilbert space of functions is isomorphic to $\mathbf{l}^2$ [7, p. 47].

⁸ A is unitary since

$$\|Af\|^2 = \langle Af, Af \rangle = \sum_n |a_n|^2 = \sum_n \left| \left\langle f, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle \right|^2 = \|f\|^2,$$

where the last equality is obtained from Parseval's equality (see Section 2.5.6) and it is clear that A^{-1} exists and is continuous.

3.2 Linear Functionals

3.2.1 General Concepts

A linear operator maps elements of one linear space into another linear space, or perhaps a space into itself. A linear functional maps elements of a linear space into $\mathbf{C}$, the set of complex numbers. We concentrate on mappings of Hilbert spaces.

Definition 3.9. A mapping l from H to $\mathbf{C}$ (denoted as $l : H \rightarrow \mathbf{C}$ or $l(H, \mathbf{C})$) is called a linear functional if, for all $x, y \in D_l$, there correspond numbers $l(x), l(y) \in \mathbf{C}$ such that

$$l(\alpha x + \beta y) = \alpha l(x) + \beta l(y),$$

where $\alpha, \beta \in \mathbf{C}$.

Under this definition ordinary linear functions $x(t)$ that map $t \in \mathbf{C}$ into $\mathbf{C}$ are linear functionals. More often the term functional is used for “functions of functions,” e.g., $l : H \rightarrow \mathbf{C}$ where perhaps $H = \mathbf{L}^2(a, b)$. As an example, for $x \in \mathbf{L}^2(a, b)$, $\int_a^b x(t) dt$ is a linear functional $l : \mathbf{L}^2(a, b) \rightarrow \mathbf{C}$.

Definition 3.10. A linear functional l is called continuous if the mapping $l : H \rightarrow \mathbf{C}$ is continuous on H .

If a linear functional l is continuous at one point, then it is continuous everywhere on its domain.

Definition 3.11. A linear functional $l : H \rightarrow \mathbf{C}$ is bounded on its domain if there exists a number $k \in \mathbf{R}$ such that for all elements $x \in D_l$,

$$\|l(x)\| = |l(x)| \leq k \|x\|.$$

As with the more general operators previously discussed, the definition of boundedness of a functional brings about the concept of the norm of a functional.

Definition 3.12. The norm of a linear functional $l : H \rightarrow \mathbf{C}$, denoted as $\|l\|$, is the smallest number k that satisfies

$$|l(x)| \leq k \|x\|$$

for all $x \in D_l$.

In parallel with operators, continuity, boundedness, domain, range, and null space of linear functionals are defined in obvious ways from the corresponding definitions for linear operators. In addition, as with linear operators, continuity and boundedness of a linear functional go hand in hand. In this way a linear functional $l : H \rightarrow \mathbf{C}$ can be thought of as a special case of a linear operator $A : H_1 \rightarrow H_2$. It can be shown that every linear functional $l : \mathbf{C}^n \rightarrow \mathbf{C}$ is bounded (see Theorem 3.5). The space of all bounded linear functionals $l : H \rightarrow \mathbf{C}$ is known as the *dual space* of H .

Riesz Representation Theorem

An important theoretical use of linear functionals is in defining *adjoint* operators. Before proceeding, note that for y a fixed element of an inner product space, it is easy to see that $l(x) = \langle x, y \rangle$ is a bounded linear functional (although $l(x) = \langle y, x \rangle$ for y a fixed element is not linear). Indeed, linearity and continuity follow directly from the definition of the inner product. Therefore, for each element of an inner product space there is a natural functional given by an inner product. In Hilbert spaces the converse is also true.

Theorem 3.6. (*Riesz representation theorem*) *Let l be a bounded linear functional on a Hilbert space H . There is a unique element $y \in H$ such that $l(x) = \langle x, y \rangle$ for all $x \in H$. Moreover, $\|l\| = \|y\|$.*

The Riesz representation theorem guarantees that every bounded linear functional on $\mathbf{L}^2(a, b)$ has the form

$$l(x) = \int_a^b x(t) \overline{y(t)} dt$$

with the norm $\|l\| = \|y\| = (\int_a^b |y(t)|^2 dt)^{1/2}$ for some $y(t) \in \mathbf{L}^2(a, b)$ and all $x(t) \in \mathbf{L}^2(a, b)$. Similarly, every bounded linear functional on $\mathbf{l}^2$ has the form

$$l(\mathbf{x}) = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

with the norm $\|l\| = \|\mathbf{y}\| = (\sum_{i=1}^{\infty} |y_i|^2)^{1/2}$ for some $\mathbf{y} = \{y_1, y_2, \dots\} \in \mathbf{l}^2$ and all $\mathbf{x} = \{x_1, x_2, \dots\} \in \mathbf{l}^2$, and every bounded linear functional on $\mathbf{C}^n$ has the form

$$l(\mathbf{x}) = \sum_{i=1}^n x_i \overline{y_i}$$

with the norm $\|l\| = \|\mathbf{y}\| = (\sum_{i=1}^n |y_i|^2)^{1/2}$ for some $\mathbf{y} = \{y_1, y_2, \dots, y_n\} \in \mathbf{C}^n$ and all $\mathbf{x} = \{x_1, x_2, \dots, x_n\} \in \mathbf{C}^n$.

3.2.2 Examples Related to Linear Functionals

1. Consider a linear operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$. A bounded linear functional on $\mathbf{L}^2(\Omega)$ is

$$l(x) = \langle Ax, y \rangle = \int_{\Omega} (Ax)(\mathbf{t}) \overline{y(\mathbf{t})} d\Omega$$

for some $y(\mathbf{t}) \in \mathbf{L}^2(\Omega)$ and all $x(\mathbf{t})$ in the domain $D_A = \{x : x(\mathbf{t}), (Ax)(\mathbf{t}) \in \mathbf{L}^2(\Omega)\}$. If the adjoint operator $A^* : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$

exists (see Section 3.3), then

$$l(x) = \langle Ax, y \rangle = \langle x, A^*y \rangle = \int_{\Omega} x(\mathbf{t}) \overline{(A^*y)(\mathbf{t})} d\Omega,$$

where $y(\mathbf{t})$ is in the domain $D_{A^*} = \{y : y(\mathbf{t}), (A^*y)(\mathbf{t}) \in \mathbf{L}^2(\Omega)\}$.

2. The linear functional $\phi_c : \mathbf{C}(a, b) \rightarrow \mathbf{C}$ defined by $\phi_c(f) \equiv f(c)$, where $c \in [a, b]$, can be written in terms of a delta-function (in the distributional sense; note $\delta \notin \mathbf{C}(\Omega), \mathbf{L}^2(\Omega)$) as

$$\phi_c(f) = \langle f, \delta \rangle = \int_a^b f(x) \delta(x - c) dx.$$

3. As an example in electromagnetics, scalar components of the magnetic vector potential A_α at some *fixed* point in space $\mathbf{r}$ caused by an electric current in free space are linear functionals of the current density J_α ,

$$A_\alpha(J_\alpha) = \left\langle \mu J_\alpha, \frac{e^{jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \right\rangle = \int_{\Omega} \mu J_\alpha(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} d\Omega',$$

as are various electric- and magnetic field components. Alternatively, for a fixed J_α , A_α can be thought of as a nonlinear functional of the permittivity function ε via the wavenumber k .

4. The voltage V_{ab} between two fixed points a, b is a linear functional of the electric field $\mathbf{E}$ via

$$V_{ab}(\mathbf{E}) = - \int_b^a \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l}.$$

As further examples, the total charge Q in a volume Ω is a linear functional of the charge density ρ via $Q(\rho) = \int_{\Omega} \rho(\mathbf{r}) d\Omega$, and electrostatic power P is a linear functional (of $\mathbf{E}$ or $\mathbf{J}$) via $P = \int_{\Omega} \mathbf{E} \cdot \mathbf{J} d\Omega$. Numerous other examples can obviously be found.

5. Consider the operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ associated with the infinite matrix $[a_{ij}]$ and defined by $(A\mathbf{x})_{i=1}^{\infty} = \sum_{j=1}^{\infty} a_{ij}x_j$. Then a linear functional $l(\mathbf{x})$ is defined on $\mathbf{l}^2$ as

$$l(\mathbf{x}) \equiv \langle A\mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}x_j \bar{y}_i$$

for some $\mathbf{y} \in \mathbf{l}^2$ and all $\mathbf{x}$ in the domain $D_A = \{\mathbf{x} : \mathbf{x}, A\mathbf{x} \in \mathbf{l}^2\}$. If the operator A is bounded assuming $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$ (Hilbert-Schmidt operator), then the functional $l(\mathbf{x})$ is bounded on $\mathbf{l}^2$.

6. Consider the vector differential operator $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)^3$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \nabla \times \nabla \times \mathbf{x}(\mathbf{t}),$$

$$D_A \equiv \left\{ \mathbf{x} : \mathbf{x}(\mathbf{t}), \nabla \times \nabla \times \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3 \right\}.$$

A linear functional $l(\mathbf{x})$ is defined by

$$l(\mathbf{x}) \equiv \langle A\mathbf{x}, \mathbf{y} \rangle = \int_{\Omega} (\nabla \times \nabla \times \mathbf{x}) \cdot \bar{\mathbf{y}} \, d\Omega$$

for some 3-tuple $\mathbf{y}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3$ and all 3-tuples $\mathbf{x}(\mathbf{t}) \in D_A$.

3.3 Adjoint Operators

3.3.1 General Concepts: Bounded Operators

The Riesz representation theorem is useful for defining in a natural way the important concept of an operator *adjoint* to a given operator. Consider first a bounded linear operator $A : H_1 \rightarrow H_2$. For every $y \in H_2$, it can be seen that $l(x) = \langle Ax, y \rangle_2$ is a bounded linear functional $l : H_1 \rightarrow \mathbb{C}$ for all $x \in H_1$. Therefore, by the Riesz representation theorem there exists a unique $y^* \in H_1$ such that for all $x \in H_1$,

$$\langle Ax, y \rangle_2 = \langle x, y^* \rangle_1.$$

Because y^* depends on y , we introduce a new operator $A^* : H_2 \rightarrow H_1$, called the *adjoint* of A , defined by $A^*y = y^*$, leading to

$$\langle Ax, y \rangle_2 = \langle x, A^*y \rangle_1.$$

Usually we will drop the subscripts on the inner products.

Therefore, if A is a bounded linear operator, by the Riesz representation theorem a unique adjoint exists. It can be shown that⁹ $\|A\| = \|A^*\|$, and so the adjoint of a bounded linear operator is a bounded. Some convenient properties relating to adjoints of bounded linear operators are the following.

- $\|A^*A\| = \|AA^*\| = \|A\|^2$
- $(A + B)^* = A^* + B^*$
- $(\alpha A)^* = \bar{\alpha}A^*$

⁹This simply follows from

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle| = \sup_{\|x\|=\|y\|=1} |\langle x, A^*y \rangle| = \|A^*\|.$$

- $(CA)^* = A^*C^*$, where $C : H_2 \rightarrow H_3$
- $(A^*)^* = A$

where in the above $A, B : H_1 \rightarrow H_2$. Relationships between the null space and range of an operator and those of its adjoint are given by the next theorem.

Theorem 3.7. *Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Then*

- $N_A = (R_{A^*})^\perp$
- $N_{A^*} = (R_A)^\perp$
- $\widetilde{R_A} = (N_{A^*})^\perp$
- $\widetilde{R_{A^*}} = (N_A)^\perp$

where $\widetilde{R_A}$ is the closure of R_A .

From Theorem 2.7 we see that since the null space is always closed, if the range is closed then the last two properties are repetitive with the first two.

3.3.2 Examples of Operator Adjoint: Bounded Operators

1. The identity and zero operators have simple adjoints; $I^* = I$, $0^* = 0$.
2. It was shown before that a bounded operator $A : C^n \rightarrow C^n$, with $\{x_1, x_2, \dots, x_n\}$ an orthonormal basis for C^n , is represented by the matrix $[a_{ij}]$, where $a_{ij} = \langle Ax_j, x_i \rangle$. From the definition $\langle Ax_j, x_i \rangle = \langle x_j, A^*x_i \rangle$ one can see that the adjoint operator $A^* : C^n \rightarrow C^n$ is represented in the basis $\{x_n\}$ by the matrix $[\overline{a_{ji}}]$, the conjugate transpose¹⁰ of the matrix $[a_{ij}]$.

The same result is obtained for the operator $A : l^2 \rightarrow l^2$ defined by $(A\mathbf{x})_{i=1}^\infty = \sum_{j=1}^\infty a_{ij}x_j$, corresponding to the infinite matrix $[a_{ij}]$ with $\sum_{i=1}^\infty \sum_{j=1}^\infty |a_{ij}|^2 < \infty$ (Hilbert–Schmidt operator).

3. The multiplication operator $A : L^2(\Omega) \rightarrow L^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv f(\mathbf{t})x(\mathbf{t})$$

is a bounded linear operator assuming $k = \max_{\mathbf{t} \in \Omega} |f(\mathbf{t})| < \infty$ where $\Omega \subset \mathbf{R}^n$. Then

$$\langle Ax, y \rangle = \int_{\Omega} f(\mathbf{t})x(\mathbf{t})\overline{y(\mathbf{t})} d\Omega = \langle x, \overline{f} y \rangle = \langle x, A^*y \rangle,$$

¹⁰Note that this is true only when $\{x_n\}$ is an orthonormal basis for C^n . When $\{x_n\}$ is merely orthogonal, then the matrix representation of A^* is given by a similarity transformation (see Section 4.3.1) of the conjugate transpose matrix $[\overline{a_{ji}}]$.

where

$$(A^*y)(\mathbf{t}) = \overline{f(\mathbf{t})}y(\mathbf{t}).$$

Therefore, for the operator “multiplication by $f(\mathbf{t})$,” the adjoint is “multiplication by $\overline{f(\mathbf{t})}$.”

4. The integral operator $A : \mathbf{L}^2(\Omega)^m \rightarrow \mathbf{L}^2(\Omega)^m$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \int_{\Omega} \underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \cdot \mathbf{x}(\mathbf{s}) d\Omega,$$

where $\Omega \subset \mathbf{R}^n$ with $\underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \in \mathbf{L}^2(\Omega \times \Omega)^{m \times m}$ (in the sense that each component of the $(m \times m)$ -tuple dyadic kernel $\underline{\mathbf{k}}(\mathbf{t}, \mathbf{s})$ belongs to $\mathbf{L}^2(\Omega \times \Omega)$), is bounded. The adjoint is found from

$$\begin{aligned} \langle A\mathbf{x}, \mathbf{y} \rangle &= \int_{\Omega} \int_{\Omega} \underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \cdot \mathbf{x}(\mathbf{s}) d\Omega_s \cdot \overline{\mathbf{y}(\mathbf{t})} d\Omega_t \\ &= \int_{\Omega} \mathbf{x}(\mathbf{s}) \cdot \overline{\int_{\Omega} \underline{\mathbf{k}}^{\top}(\mathbf{t}, \mathbf{s}) \cdot \mathbf{y}(\mathbf{t}) d\Omega_t} d\Omega_s = \langle \mathbf{x}, A^*\mathbf{y} \rangle \end{aligned}$$

such that the adjoint operator is

$$(A^*\mathbf{y})(\mathbf{t}) = \int_{\Omega} \overline{\underline{\mathbf{k}}^{\top}(\mathbf{s}, \mathbf{t})} \cdot \mathbf{y}(\mathbf{s}) d\Omega$$

with the adjoint kernel $\underline{\mathbf{k}}^*(\mathbf{t}, \mathbf{s}) = \overline{\underline{\mathbf{k}}^{\top}(\mathbf{s}, \mathbf{t})}$, the conjugate of the transpose of the original kernel with the variables interchanged. For example, the dyadic Green's function for static fields

$$\underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{I}} \frac{1}{4\pi R} = \underline{\mathbf{k}}(\mathbf{r}, \mathbf{r}')$$

(with $R = |\mathbf{r} - \mathbf{r}'|$) provides a self-adjoint (see Section 3.4) bounded kernel in $\mathbf{R}^n$, $n \geq 2$.

Similarly, the dyadic Green's function

$$\underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{I}} \frac{e^{-jkR}}{4\pi R} = \underline{\mathbf{k}}(\mathbf{r}, \mathbf{r}')$$

has adjoint

$$\underline{\mathbf{k}}^*(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{I}} \frac{e^{+jkR}}{4\pi R}.$$

In the scalar case of $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} k(\mathbf{t}, \mathbf{s})x(\mathbf{s}) d\Omega,$$

the adjoint operator is

$$(A^*y)(\mathbf{t}) = \int_{\Omega} \overline{k(\mathbf{s}, \mathbf{t})} y(\mathbf{s}) d\Omega = \int_{\Omega} k^*(\mathbf{t}, \mathbf{s}) y(\mathbf{s}) d\Omega$$

with the adjoint kernel $k^*(\mathbf{t}, \mathbf{s}) = \overline{k(\mathbf{s}, \mathbf{t})}$, the conjugate of the original kernel with the variables interchanged.

5. The integration operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv \int_a^t x(s) ds$$

is a linear bounded operator. Its adjoint is found as a special case of the previous example ($\Omega = (a, b)$) with $k(t, s) = H(t - s)$. Therefore, the adjoint operator is $k^*(t, s) = H(s - t)$, leading to

$$(A^*y)(t) = \int_t^b y(s) ds.$$

3.3.3 General Concepts: Unbounded Operators

Before considering the adjoint of the differentiation operator, we need to investigate the concept of an adjoint for unbounded operators. Since the Riesz representation theorem no longer holds, we are no longer guaranteed that for an element $y \in H_2$ an element $y^* \in H_1$ exists such that $\langle Ax, y \rangle = \langle x, y^* \rangle$ for all $x \in H_1$. Often, though, such an element exists, and we obtain the adjoint A^* as before; $A^*y = y^*$ such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$. In general, such an adjoint may not be unique. If the operator is densely defined, which is the case of primary interest here, the adjoint operator is unique.¹¹

Actually, in some cases we need to be more careful. If A is bounded, then $\langle Ax, y \rangle = \langle x, A^*y \rangle$ actually *defines* the adjoint. If A is unbounded, then this relationship merely describes the “action” of the operator but does not always lead to the correct identification of the domain D_{A^*} . For an unbounded, densely defined operator, a rigorous definition of the adjoint is the following.

Definition 3.13. *Let $A : H \rightarrow H$ be a densely defined operator. Then*

$$D_{A^*} \equiv \{z \in H : \exists y \in H \text{ such that } \langle Ax, z \rangle = \langle x, y \rangle, \forall x \in D_A\},$$

$$A^*z \equiv y.$$

¹¹Indeed, if $\langle Ax, y \rangle = \langle x, y^* \rangle$ were true for two different y^* , say y_1^* and y_2^* , then $\langle x, y_1^* - y_2^* \rangle = 0$ for all $x \in D_A$. Because D_A is dense in H_1 , only the zero element is orthogonal to every element in D_A (by Theorem 2.7(c)); hence $y_1^* = y_2^*$.

However, in most practical cases the simpler procedure described below leads to the correct domain of A^* .

In summary, all bounded linear operators have a unique adjoint, and for all unbounded, densely defined linear operators that possess an adjoint, the adjoint is unique. All unbounded differential operators considered in this text possess adjoints.

3.3.4 Examples of Operator Adjoints: Unbounded Operators

1. Consider the differentiation operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$\begin{aligned} (Ax)(t) &\equiv \frac{d}{dt}x(t) = x'(t), \\ D_A &\equiv \{x : x(t), x'(t) \in \mathbf{L}^2(a, b), x(a) = 0\}. \end{aligned}$$

The adjoint operator can be found by integration by parts:

$$\begin{aligned} \langle Ax, y \rangle &= \int_a^b x'(t) \overline{y(t)} dt \\ &= x(b) \overline{y(b)} - x(a) \overline{y(a)} - \int_a^b x(t) \overline{y'(t)} dt \\ &= \langle x, A^*y \rangle. \end{aligned}$$

Therefore, the adjoint differentiation operator is

$$\begin{aligned} (A^*y)(t) &= -\frac{d}{dt}y(t), \\ D_{A^*} &= \{y : y(t), y'(t) \in \mathbf{L}^2(a, b), y(b) = 0\}; \end{aligned}$$

hence $D_{A^*} \neq D_A$ and $A^*x = -Ax$ for all $x \in D_A \cap D_{A^*}$. Note that we actually get a conjugate-adjoint boundary condition on $\overline{y}$, which is easily converted into an adjoint boundary condition on y .

2. In the previous example, if we define

$$D_A \equiv \{x : x(t), x'(t) \in \mathbf{L}^2(a, b), x(a) = x(b) = 0\},$$

then $A^*x = -Ax$ for all suitable x , but no boundary conditions result to restrict D_{A^*} , i.e.,

$$D_{A^*} = \{y : y(t), y'(t) \in \mathbf{L}^2(a, b)\} \supset D_A.$$

Furthermore, if D_A is such that no boundary conditions are imposed on x , then boundary conditions $y(a) = y(b) = 0$ will be induced on the adjoint.

3. Consider the differentiation operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv \frac{d^2}{dt^2}x(t) = x''(t),$$

$$D_A \equiv \{x : x(t), x'(t), x''(t) \in \mathbf{L}^2(a, b), x(a) = x(b) = 0\}.$$

This can be thought of as the one-dimensional Laplacian operator. The adjoint operator can be found by integration by parts twice (equivalently Green's second theorem)

$$\begin{aligned} \langle Ax, y \rangle &= \int_a^b \frac{d^2}{dt^2}x(t) \overline{y(t)} dt \\ &= \left(\frac{d}{dt}x(t) \right) \overline{y(t)} \Big|_a^b - x(t) \frac{d}{dt} \overline{y(t)} \Big|_a^b + \int_a^b x(t) \frac{d^2}{dt^2} \overline{y(t)} dt \\ &= \langle x, A^*y \rangle. \end{aligned} \tag{3.2}$$

Therefore,

$$(A^*y)(t) = \frac{d^2}{dt^2}y(t) = (Ay)(t),$$

$$D_{A^*} = \{y : y(t), y'(t), y''(t) \in \mathbf{L}^2(a, b), y(a) = y(b) = 0\} = D_A$$

such that $A^* = A$.

4. Similar to the last example, the Laplacian operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \nabla^2 x(\mathbf{t}),$$

$$D_A \equiv \{x : x(\mathbf{t}), \nabla^2 x(\mathbf{t}) \in \mathbf{L}^2(\Omega), x(\mathbf{t})|_{\Gamma} = 0\},$$

where Γ is the smooth boundary of $\Omega \subset \mathbf{R}^n$, is such that $A^* = A$. Indeed, from Green's second theorem

$$\begin{aligned} \langle Ax, y \rangle &= \int_{\Omega} \overline{y} \nabla^2 x d\Omega \\ &= \oint_{\Gamma} \left(\overline{y} \frac{\partial x}{\partial n} - x \frac{\partial \overline{y}}{\partial n} \right) dS + \int_{\Omega} x \nabla^2 \overline{y} d\Omega \\ &= \langle x, A^*y \rangle. \end{aligned}$$

5. Consider the vector differential operator $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)^3$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \nabla \times \nabla \times \mathbf{x}(\mathbf{t}),$$

$$D_A \equiv \{\mathbf{x} : \mathbf{x}(\mathbf{t}), \nabla \times \nabla \times \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3, \mathbf{n} \times \mathbf{x}(\mathbf{t})|_{\Gamma} = \mathbf{0}\}.$$

The adjoint operator can be found using the second vector Green's theorem

$$\begin{aligned}
 \langle A\mathbf{x}, \mathbf{y} \rangle &= \int_{\Omega} \bar{\mathbf{y}} \cdot (\nabla \times \nabla \times \mathbf{x}) \, d\Omega \\
 &= \oint_{\Gamma} ((\mathbf{n} \times \mathbf{x}) \cdot (\nabla \times \bar{\mathbf{y}}) - (\mathbf{n} \times \bar{\mathbf{y}}) \cdot (\nabla \times \mathbf{x})) \, dS \\
 &\quad + \int_{\Omega} \mathbf{x} \cdot (\nabla \times \nabla \times \bar{\mathbf{y}}) \, d\Omega \\
 &= \langle \mathbf{x}, A^* \mathbf{y} \rangle,
 \end{aligned}$$

where

$$\begin{aligned}
 (A^* \mathbf{y})(\mathbf{t}) &= \nabla \times \nabla \times \mathbf{y}(\mathbf{t}) = (A\mathbf{y})(\mathbf{t}), \\
 D_{A^*} &= \left\{ \mathbf{y} : \mathbf{y}(\mathbf{t}), \nabla \times \nabla \times \mathbf{y}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3, \mathbf{n} \times \mathbf{y}(\mathbf{t})|_{\Gamma} = \mathbf{0} \right\} \\
 &= D_A.
 \end{aligned}$$

Thus, $A^* = A$. This example can be generalized for a (3×3) -tuple dyadic function (rank 2 tensor) $\underline{\mathbf{y}}(\mathbf{t})$ using the second vector-dyadic Green's theorem as follows,

$$\begin{aligned}
 \langle A\mathbf{x}, \underline{\mathbf{y}} \rangle &= \int_{\Omega} (\nabla \times \nabla \times \mathbf{x}) \cdot \underline{\mathbf{y}} \, d\Omega \\
 &= \oint_{\Gamma} ((\mathbf{n} \times \mathbf{x}) \cdot (\nabla \times \underline{\mathbf{y}}) - (\nabla \times \mathbf{x}) \cdot (\mathbf{n} \times \underline{\mathbf{y}})) \, dS \\
 &\quad + \int_{\Omega} \mathbf{x} \cdot (\nabla \times \nabla \times \underline{\mathbf{y}}) \, d\Omega \\
 &= \langle \mathbf{x}, A^* \underline{\mathbf{y}} \rangle,
 \end{aligned}$$

where again we find $A^* = A$. We understand $\underline{\mathbf{y}}(\mathbf{t}), \nabla \times \nabla \times \underline{\mathbf{y}}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^{3 \times 3}$ in the sense that components of the dyadic function and components of the differential operator belong to $\mathbf{L}^2(\Omega)$, and that $\mathbf{L}^2(\Omega)^{3 \times 3}$ represents a direct sum of spaces $\mathbf{L}^2(\Omega)$.

6. Consider the vector differential operator (curl) $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)^3$ defined by

$$\begin{aligned}
 (A\mathbf{x})(\mathbf{t}) &\equiv \nabla \times \mathbf{x}(\mathbf{t}), \\
 D_A &\equiv \left\{ \mathbf{x} : \mathbf{x}(\mathbf{t}), \nabla \times \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3, \mathbf{n} \times \mathbf{x}(\mathbf{t})|_{\Gamma} = \mathbf{0} \right\},
 \end{aligned}$$

where D_A is a subspace of $H(\text{curl}, \Omega)$. The adjoint operator can be found using the vector identity $\nabla \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot \nabla \times \mathbf{x} - \mathbf{x} \cdot \nabla \times \mathbf{y}$ and the divergence theorem as

$$\begin{aligned}
 (A^* \mathbf{y})(\mathbf{t}) &= \nabla \times \mathbf{y}(\mathbf{t}), \\
 D_{A^*} &= \left\{ \mathbf{y} : \mathbf{y}(\mathbf{t}), \nabla \times \mathbf{y}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3 \right\}.
 \end{aligned}$$

In a similar manner, the adjoint of the differential operator (divergence) $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \nabla \cdot \mathbf{x}(\mathbf{t}),$$

$$D_A \equiv \left\{ \mathbf{x} : \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3, \nabla \cdot \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega), \mathbf{n} \cdot \mathbf{x}(\mathbf{t})|_{\Gamma} = 0 \right\},$$

where D_A is a subspace of $H(\text{div}, \Omega)$, is found from the vector identity $\nabla \cdot (\psi \mathbf{x}) = \mathbf{x} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{x}$ and the divergence theorem as

$$(A^* \psi)(\mathbf{t}) = -\nabla \psi(\mathbf{t}),$$

$$D_{A^*} = \left\{ \psi : \psi(\mathbf{t}), \nabla \psi(\mathbf{t}) \in \mathbf{L}^2(\Omega) \right\}.$$

Therefore, the formal adjoint¹² of curl is curl, the formal adjoint of divergence is negative gradient, and the formal adjoint of gradient is negative divergence.

General Procedure for Determining Operator Adjoint

It is worthwhile to summarize the procedure for obtaining the adjoint of a differential operator. Let L be a general differential operator (scalar, vector, or dyadic) with domain D_L ; the specification of D_L may include boundary conditions $B(x) = \eta$, $x \in D_L$. One applies an appropriate Green's second theorem, or related vector identity corresponding to integration by parts, to the pair $x, \bar{y}$, where y is conjugated to account for the conjugate in the inner product. The domain of the adjoint is determined by requiring that the "integrated terms" in the Green's theorem (i.e., the boundary terms) vanish when the homogeneous boundary condition $B(x) = 0$ is enforced (regularity of the functions in D_{L^*} is assumed). This homogeneous condition is applied even if the given boundary conditions are inhomogeneous, i.e., if $\eta \neq 0$ [8, pp. 180, 185], [9, pp. 72–73]. This leads to conjugate-adjoint boundary conditions $B_{ca}^*(\bar{y}) = 0$, which are then converted to adjoint boundary conditions B^* on y , $B^*(y) = \overline{B_{ca}^*(\bar{y})} = 0$ (adjoint boundary conditions are always homogeneous). The adjoint operator is then easily recognized from the remaining terms. In a real space the conjugate is omitted.

To generalize this for any linear operator L consider the expression

$$\int_{\Omega} (Lx) \bar{y} d\Omega = \int_{\Omega} x \overline{(L^*y)} d\Omega + \int_{\Gamma} J(x, \bar{y}) d\Gamma, \quad (3.3)$$

which is known as a *generalized Green's theorem* [10, p. 870] (for $n = 1$, Ω is a line segment and the boundary term is replaced with a term

¹²The formal adjoint is the adjoint without considering the associated domains, i.e., concentrating on the formal action of the operator (see Section 3.4).

involving the two endpoints of the line). The term involving J is known as the *bilinear concomitant* or the *conjunct*, and the operator L^* is called the *formal adjoint* of L . The domain of L^* is determined by requiring $\int_{\Gamma} J(x, \bar{y}) d\Gamma = 0$ for $x \in D_L$ (again, if $B(x) = \eta \neq 0$, we enforce $B(x) = 0$ in the conjunct) leading to conjugate-adjoint boundary conditions on $\bar{y}$, which are then converted to adjoint conditions on $y \in D_{L^*}$. With the conjunct so removed we have $\langle Lx, y \rangle = \langle x, L^*y \rangle$ as expected. Note that the conjunct is simply the “integrated term” in an n -dimensional integration-by-parts procedure.

If L^* is a real operator,¹³ as is often the case, then the generalized Green’s theorem becomes

$$\int_{\Omega} (Lx) \bar{y} d\Omega = \int_{\Omega} x (L^*\bar{y}) d\Omega + \int_{\Gamma} J(x, \bar{y}) d\Gamma. \quad (3.4)$$

This form can represent any of the Green’s second theorems listed in Appendix A.4. For example, if $L = \nabla^2 = L^*$ (formally) and $J(x, \bar{y}) = \bar{y} dx/dn - x d\bar{y}/dn$ where d/dn is the outward normal derivative on Γ , we then have the standard Green’s second theorem for scalars (substitute $h = \bar{y}$ to get the usual form without conjugation). In one dimension, $L = d/dt = -L^*$ and $J(x, \bar{y}) = x\bar{y}$ lead to the usual integration by parts formula. In the vector case (see, e.g., [11, Ch. III] for related material) (3.4) becomes

$$\int_{\Omega} (\mathbf{L}\mathbf{x}) \cdot \bar{\mathbf{y}} d\Omega = \int_{\Omega} \mathbf{x} \cdot (\mathbf{L}^*\bar{\mathbf{y}}) d\Omega + \int_{\Gamma} \mathbf{J}(\mathbf{x}, \bar{\mathbf{y}}) \cdot d\Gamma. \quad (3.5)$$

For $\mathbf{L} = \nabla \times \nabla \times = \mathbf{L}^*$ and $\mathbf{J}(\mathbf{x}, \bar{\mathbf{y}}) = \mathbf{x} \times \nabla \times \bar{\mathbf{y}} + (\nabla \times \mathbf{x}) \times \bar{\mathbf{y}}$, one has the usual vector second Green’s theorem.

3.4 Self-Adjoint, Symmetric, Normal, and Unitary Operators

3.4.1 General Concepts

In discussing self-adjoint operators we will naturally restrict our attention to linear operators $A : H \rightarrow H$. We include operators acting on H and on domains dense in H .

Self-Adjoint Operators

Definition 3.14. An operator A is called *self-adjoint* if $A = A^*$.

Note by this we mean $D_A = D_{A^*}$ and $A^*x = Ax$ for all $x \in D_A = D_{A^*}$. That is, the operators A and A^* act on the same set of elements of H , and

¹³An operator L is called *real* if $\overline{Lx} = L\bar{x}$.

the action of A and A^* on those elements is identical. Such operators are also called *Hermitian*, a name usually reserved for the finite-dimensional (matrix) case.

Obviously, for self-adjoint operators $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in D_A = D_{A^*}$. Moreover, self-adjoint operators belong to a larger class of *symmetric* operators. In the following we focus on densely defined operators, since for this class of operator if an adjoint exists it is unique.

Symmetric Operators

Definition 3.15. *A densely defined operator A is called symmetric (or formally self-adjoint) if $Ax = A^*x$ for all $x \in D_A \subseteq D_{A^*}$.*

This is equivalent to $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in D_A$. It turns out that for bounded operators the difference between symmetry and self-adjointness is not important; however, this is not the case for unbounded operators. To see this we need to introduce the concept of an *operator extension*.

Definition 3.16. *A linear operator B is said to be the extension of the linear operator A if $D_A \subseteq D_B$ and $Ax = Bx$ for all $x \in D_A$.*

It can be shown that if a bounded operator is not defined *on* a Hilbert space, but is defined on a linear subspace *in* a Hilbert space, it can always be extended to the whole space (see, e.g., [12, pp. 325–326]). For most purposes then we may take $D_A = H$ for bounded operators $A : H \rightarrow H$, in which case the operator always acts *on* the Hilbert space and $D_{A^*} = H$. Therefore, we conclude a bounded operator A is self-adjoint (at least under an extension) if it is symmetric.

It is clear that every self-adjoint operator is symmetric and that every bounded symmetric operator is self-adjoint. It is possible for an unbounded, symmetric operator to be nonself-adjoint, even if it is densely defined, as the examples in Section 3.3.4 illustrate.

Since in general unbounded operators cannot be extended to the whole space, even if they are densely defined, then for unbounded operators symmetry does not imply self-adjointness. With self-adjointness being a much stronger condition than mere symmetry, it is fortunate that many differential operators of interest in electromagnetics are not only symmetric but are in fact self-adjoint (dynamic problems in lossless unbounded regions being a notable exception, since radiation conditions don't lead to a self-adjoint operator).

Some basic properties relating to self-adjoint operators are given below.

Theorem 3.8. *If $A : H \rightarrow H$ is a bounded linear operator, then A^*A , AA^* , and $A + A^*$ are self-adjoint.*

Proof. $(A^*A)^* = A^*A^{**} = A^*A$, $(AA^*)^* = A^{**}A^* = AA^*$, and $(A + A^*)^* = A^* + A^{**} = A^* + A = A + A^*$. ■

Theorem 3.9. *Let $A : H \rightarrow H$ be a bounded linear operator. Then there exist unique self-adjoint operators S and T such that $A = S + iT$ and $A^* = S - iT$.*

In particular,

$$\begin{aligned} S &= (A + A^*)/2, \\ T &= (A - A^*)/2i, \end{aligned}$$

a property used in the method of characteristics [13]. To prove that S and T are self-adjoint operators, we use the properties of adjoint operators:

$$\begin{aligned} S^* &= \left(\frac{A + A^*}{2} \right)^* = \frac{A^* + A^{**}}{2} = \frac{A + A^*}{2} = S, \\ T^* &= \left(\frac{A - A^*}{2i} \right)^* = \frac{A^* - A^{**}}{-2i} = \frac{A - A^*}{2i} = T. \end{aligned}$$

Note that if A is self-adjoint, then $S = A$ and $T = 0$.

Theorem 3.10. *Let $A, B : H \rightarrow H$ be bounded self-adjoint operators. The product operator $AB : H \rightarrow H$ is self-adjoint if and only if the operators commute, i.e., $AB = BA$.*

The fact that $\langle Ax, x \rangle \in \mathbf{R}$ for a self-adjoint linear operator leads to a special characterization of its norm (it is easy to see that for H a complex Hilbert space, if $\langle Ax, x \rangle \in \mathbf{R}$ then A is symmetric, and conversely if A is symmetric then $\langle Ax, x \rangle \in \mathbf{R}$).

Theorem 3.11. *Let $A : H \rightarrow H$ be a bounded linear self-adjoint operator. Then*

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

Finally, we introduce the idea of a *normal* operator and a *unitary* operator.

Normal Operators

Definition 3.17. *A bounded linear operator $A : H \rightarrow H$ is said to be normal if*

$$AA^* = A^*A.$$

We introduce

$$[A, B] \equiv AB - BA,$$

called the *commutator* of operators A and B [6].¹⁴ Then the operator A is normal if $[A, A^*] = AA^* - A^*A = 0$.

Theorem 3.12. *A bounded linear operator $A : H \rightarrow H$ is normal if and only if $[S, T] = 0$ (S, T defined above).*

Proof. First assume A is normal. Then

$$[S, T] = \left[\frac{A + A^*}{2}, \frac{A - A^*}{2i} \right] = \frac{A^*A - AA^*}{2i} = 0.$$

Now assume $[S, T] = 0$. Then $[A, A^*] = 0$ by the equation above, hence A is normal. ■

Theorem 3.13. *A bounded linear operator $A : H \rightarrow H$ is normal if and only if $\|Ax\| = \|A^*x\|$ for all $x \in H$.*

Proof. First assume A is normal. Then for all $x \in H$, $\langle AA^*x, x \rangle = \langle A^*x, A^*x \rangle$. Also, $\langle AA^*x, x \rangle = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle$. Therefore, $\|A^*x\| = \|Ax\|$. Now assume $\|A^*x\| = \|Ax\|$ for all $x \in H$ and show A is normal. Indeed, from the above one immediately gets $\langle (AA^* - A^*A)x, x \rangle = 0$ for all $x \in H$. Since it can be shown that if $\langle Bx, x \rangle = 0$ for all x in a complex Hilbert space H , then $B = 0$, one obtains $AA^* = A^*A$ as desired. ■

Note that if $A : H \rightarrow H$ is self-adjoint then it is normal, although the converse is not generally true.

Unitary Operators

Definition 3.18. *Let $A : H \rightarrow H$ be a bounded linear operator. The operator A is said to be an unitary operator if and only if*

$$AA^* = A^*A = I,$$

i.e., $A^ = A^{-1}$.*

Obviously, unitary operators are normal. It is easy to see that if A is unitary, then $\|A\| = 1$. Furthermore,

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle = \langle x, Ix \rangle = \|x\|^2,$$

which, using polarization, is shown to also give

$$\langle Ax, Ay \rangle = \langle x, y \rangle$$

¹⁴Operators A and B for which $[A, B] = 0$ are called *commuting operators*.

for all $x, y \in H$. That is, unitary operators preserve size, distances, and angles (more generally unitary operators preserve topological structure). The range R_A of the unitary operator $A : H \rightarrow H$ coincides with that of H in a finite-dimensional space; however, this is not necessarily true in an infinite-dimensional Hilbert space.

Examples

1. The operator $M : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$, which multiplies $f \in \mathbf{L}^2(a, b)$ by $e^{i\omega t}$, where $\omega \in \mathbf{R}$, is a unitary operator.¹⁵ In essence, this multiplication “rotates” the function f , since if $f = |f| e^{i\theta_f}$, then $e^{i\omega t} f = |f| e^{i(\theta_f + \omega t)}$.
2. The Fourier transformation operator $F : \mathbf{L}^2(-\infty, \infty) \rightarrow \mathbf{L}^2(-\infty, \infty)$ defined by

$$F(f)(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

where $-\infty < k < \infty$, is a one-to-one, onto, unitary operator.

3. The operator $A : \mathbf{C}^2 \rightarrow \mathbf{C}^2$, which rotates $\mathbf{x} = (x_1, x_2) \in \mathbf{C}^2$ by the angle θ to produce $\mathbf{y} = (y_1, y_2) \in \mathbf{C}^2$, is a unitary operator with matrix representation (in the standard basis)

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

Figure 3.1 depicts the interrelationships among some of the operators discussed here. Self-adjoint operators may be further classified as to their definiteness, as discussed in the next section.

3.4.2 Examples Relating to Self-Adjointness

1. The identity ($Ix = x$) and zero ($0x = 0$) operators are self-adjoint.
2. The bounded operator $A : H \rightarrow H$, represented by the matrix $[a_{ij}]$, where $a_{ij} = \langle Ax_j, x_i \rangle$ and $\{x_1, x_2, \dots\}$ is an orthonormal basis for H , is self-adjoint if and only if $[a_{ij}] = [\overline{a_{ji}}]$, i.e., the matrix representation of the operator is Hermitian.
3. The multiplication operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv f(\mathbf{t})x(\mathbf{t})$$

is self-adjoint if and only if $f(\mathbf{t}) = \overline{f(\mathbf{t})}$, i.e., $f(\mathbf{t})$ is real-valued.

¹⁵Indeed, $\|e^{i\omega t} f\|^2 = \int |e^{i\omega t} f|^2 dt = \int |f|^2 dt = \|f\|^2$.

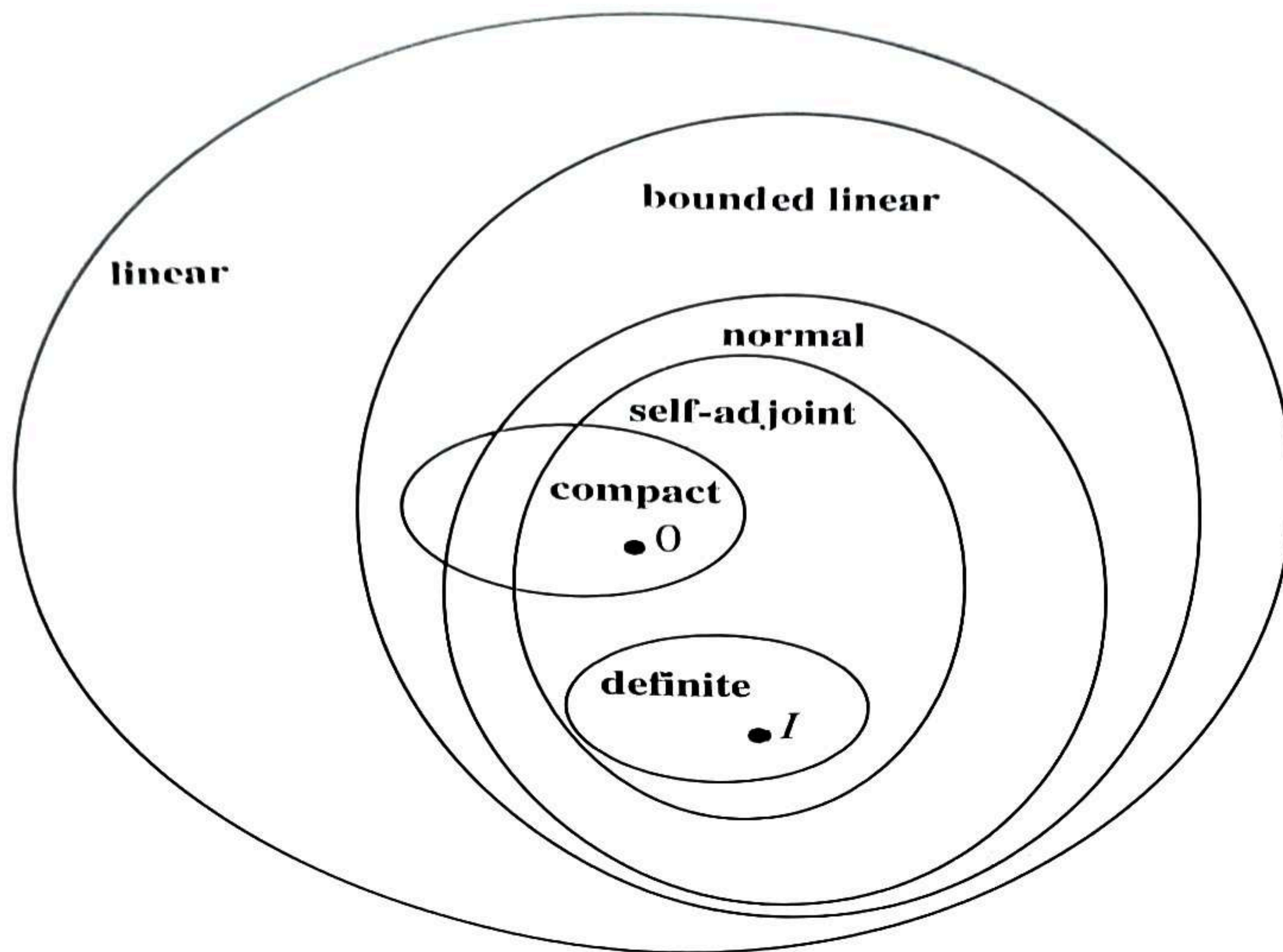


Figure 3.1: Depiction of the interrelationships among various classes of linear operators on an infinite-dimensional space. The zero and identity operators are represented by 0 and I , respectively.

4. The bounded integral operator $A : \mathbf{L}^2(\Omega)^m \rightarrow \mathbf{L}^2(\Omega)^m$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \int_{\Omega} \underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \cdot \mathbf{x}(\mathbf{s}) d\Omega,$$

where $\underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \in \mathbf{L}^2(\Omega \times \Omega)^{m \times m}$, is self-adjoint if and only if $\underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) = \underline{\mathbf{k}}^T(\mathbf{s}, \mathbf{t})$.

As a special case, the bounded integral operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} k(\mathbf{t}, \mathbf{s})x(\mathbf{s}) d\Omega,$$

where $k(\mathbf{t}, \mathbf{s}) \in \mathbf{L}^2(\Omega \times \Omega)$ (Hilbert–Schmidt kernel), is self-adjoint if and only if $k(\mathbf{t}, \mathbf{s}) = k(\mathbf{s}, \mathbf{t})$. Every bounded self-adjoint operator in $\mathbf{L}^2(\Omega)$ can be represented as an integral operator, with perhaps the kernel $k(\mathbf{t}, \mathbf{s})$ considered in the sense of a generalized function.

5. The integration operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by $(Ax)(t) \equiv \int_a^t x(s) ds$ is a linear bounded operator, but it is not self-adjoint.
6. The differentiation operator

$$(Ax)(t) \equiv \frac{d}{dt}x(t) = x'(t),$$

$$D_A \equiv \{x : x(t), x'(t) \in \mathbf{L}^2(a, b), x(a) = 0\}$$

is not self-adjoint since

$$(A^*x)(t) = -\frac{d}{dt}x(t),$$

$$D_{A^*} = \{x : x(t), x'(t) \in \mathbf{L}^2(a, b), x(b) = 0\}.$$

7. The differentiation operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv \frac{d^2}{dt^2}x(t) = x''(t),$$

$$D_A \equiv \{x : x(t), x'(t), x''(t) \in \mathbf{L}^2(a, b), x(a) = x(b) = 0\}$$

(one-dimensional Laplacian) is self-adjoint as shown in the previous section.

8. The Laplacian operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \nabla^2 x(\mathbf{t}),$$

$$D_A \equiv \{x : x(\mathbf{t}), \nabla^2 x(\mathbf{t}) \in \mathbf{L}^2(\Omega), x(\mathbf{t})|_{\Gamma} = 0\}$$

is self-adjoint as shown in the previous section.

9. The vector differential operator $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)^3$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \nabla \times \nabla \times \mathbf{x}(\mathbf{t}),$$

$$D_A \equiv \{\mathbf{x} : \mathbf{x}(\mathbf{t}), \nabla \times \nabla \times \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3, \mathbf{n} \times \mathbf{x}(\mathbf{t})|_{\Gamma} = \mathbf{0}\}$$

is self-adjoint as shown in the previous section. The generalization to a dyadic function gives a self-adjoint operator as well.

10. From the previous section we see that curl, divergence, and gradient are not self-adjoint, although curl is formally self-adjoint.
11. Electrodynamic problems involving unbounded lossless regions often lead to nonself-adjoint operators. For instance, let the Helmholtz operator A be defined by

$$Ax \equiv (\nabla^2 + k^2)x,$$

$$D_A \equiv \left\{ x : x \in \mathbf{C}^2(\mathbf{R}^3), \lim_{r \rightarrow \infty} r \left[\frac{\partial x}{\partial r} + ikx \right] = 0 \right\},$$

i.e., functions in the domain of A satisfy the outgoing radiation condition

$$\lim_{r \rightarrow \infty} r \left[\frac{\partial x}{\partial r} + ikx \right] = 0, \quad (3.6)$$

where we assume $k \in \mathbf{R}$. It is easy to see that the operator A is not self-adjoint. Indeed, from Green's second theorem,

$$\begin{aligned} \langle Ax, y \rangle &= \int_{\mathbf{R}^3} \bar{y} (\nabla^2 x + k^2 x) d\Omega \\ &= \int_{\mathbf{R}^3} x (\nabla^2 \bar{y} + k^2 \bar{y}) d\Omega + \oint_{\Gamma_\infty} \left(\bar{y} \frac{\partial x}{\partial n} - x \frac{\partial \bar{y}}{\partial n} \right) d\Gamma = \langle x, Ay \rangle \end{aligned}$$

with the last equality being valid if the boundary integral (conjunct) vanishes. With x satisfying the outgoing radiation condition (3.6), the boundary integral over Γ_∞ will vanish if y satisfies the incoming radiation condition

$$\lim_{r \rightarrow \infty} r \left[\frac{\partial y}{\partial r} - iky \right] = 0. \quad (3.7)$$

Therefore, the operator A is symmetric (formally self-adjoint) but not self-adjoint, with functions in the domain of the adjoint obeying an incoming radiation condition which is conjugate to (3.6).

3.4.3 Pseudo Adjoints, Pseudo Symmetry, and Reciprocity

Pseudo Adjoints

In many electromagnetics problems the most natural form resembling an inner product is the pseudo inner product $\langle x, y \rangle_p$, described by (2.34). Recall that unless the underlying space is real, this is not an actual inner product, and it does not generate a normed space or associated completeness properties. Nevertheless, we can introduce a *pseudo-adjoint operator* $A^{p*} : X_2 \rightarrow X_1$ satisfying

$$\langle Ax, y \rangle_{p,2} = \langle x, A^{p*}y \rangle_{p,1} \quad (3.8)$$

for the operator $A : X_1 \rightarrow X_2$ where $X_{1,2}$ are pseudo inner product spaces (see Section 2.5.2). If $X_{1,2}$ are real spaces, then the pseudo inner product is the same as the inner product, and $A^{p*} = A^*$.

As with inner product spaces, we can consider an operator $A : X_1 \rightarrow X_1$ to be *formally pseudo self-adjoint* or *pseudo symmetric* if

$$\langle Ax, y \rangle_p = \langle x, Ay \rangle_p \quad (3.9)$$

for all $x, y \in D_A$, and *pseudo self-adjoint* if, in addition, $D_{A^{p*}} = D_A$.

As an important electromagnetic example, consider the integral operator (see Section 1.3.4) defined by

$$(Ax)(\mathbf{r}) \equiv \int_{\Omega} k(\mathbf{r}, \mathbf{r}') x(\mathbf{r}') d\Omega'.$$

To determine the pseudo-adjoint operator, consider

$$\begin{aligned}\langle Ax, y \rangle_p &= \int_{\Omega} \int_{\Omega} k(\mathbf{r}, \mathbf{r}') x(\mathbf{r}') d\Omega' y(\mathbf{r}) d\Omega \\ &= \int_{\Omega} x(\mathbf{r}') \int_{\Omega} k(\mathbf{r}, \mathbf{r}') y(\mathbf{r}) d\Omega d\Omega' = \langle x, A^{p*} y \rangle_p\end{aligned}$$

(we assume that the order of integration may be interchanged, justified by properties of the kernel). Therefore, the pseudo-adjoint operator is found to be

$$(A^{p*} y)(\mathbf{r}) = \int_{\Omega} k(\mathbf{r}', \mathbf{r}) y(\mathbf{r}') d\Omega' = \int_{\Omega} k^{p*}(\mathbf{r}, \mathbf{r}') y(\mathbf{r}') d\Omega',$$

and the pseudo-adjoint kernel is

$$k^{p*}(\mathbf{r}, \mathbf{r}') = k(\mathbf{r}', \mathbf{r})$$

(recall from the examples in Section 3.3.2 that if we were using an actual inner product we would obtain the adjoint kernel as $k^*(\mathbf{r}, \mathbf{r}') = \overline{k(\mathbf{r}', \mathbf{r})}$).

For example, consider the scalar, three-dimensional free-space Green's function (1.41),

$$g(\mathbf{r}, \mathbf{r}') = \frac{e^{-j k |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|} = g(\mathbf{r}', \mathbf{r}) = g^{p*}(\mathbf{r}', \mathbf{r}).$$

For $\mathbf{r}, \mathbf{r}' \in \Omega \subset \mathbf{R}^3$ we see that integral operators of the form

$$(Af)(\mathbf{r}) = \int_{\Omega} g(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') d\Omega'$$

are pseudo self-adjoint, with bounded (weakly singular and so compact) kernel $g(\mathbf{r}, \mathbf{r}')$. The Green's function kernel represents an outgoing spherical wave, as does its pseudo adjoint, while the adjoint kernel

$$\overline{g(\mathbf{r}, \mathbf{r}')} = \frac{e^{+j k |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

represents an incoming spherical wave, where we assume $k \in \mathbf{R}$.

Referring to Example 4 in Section 3.3.2, and using reciprocity of the free-space dyadic Green's function¹⁶ $\underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ (weakly singular and so compact for the vector potential (1.72), and singular for the electric field (1.79)),

¹⁶Various free-space dyadic Green's functions are of interest (e.g., for vector potentials and for electric and magnetic fields). In all cases we have $\underline{\mathbf{G}}^T(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{G}}(\mathbf{r}', \mathbf{r})$. This also holds for any inhomogeneous reciprocal medium where $\underline{\mathbf{G}}$ directly relates the field quantity and the source, e.g.,

$$\mathbf{E}(\mathbf{r}) = \int_{\Omega} \underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') d\Omega'.$$

we see that the dyadic Green's function $\underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is a pseudo self-adjoint kernel of the pseudo self-adjoint operator

$$(Af)(\mathbf{r}) = \int_{\Omega} \underline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{f}(\mathbf{r}') d\Omega'.$$

While the concept of a pseudo inner product is very convenient in electromagnetic problems, for the complex linear spaces of primary interest it does not generally lead to the wide array of powerful analysis tools that apply to inner product spaces. In particular, the pseudo inner product does not lead to Hilbert spaces. This is also seen for the matrix representation of operators on finite-dimensional spaces.¹⁷ However, pseudo self-adjointness is sufficient to prove several important results (e.g., the stationary nature of secondary functionals, etc.), and on real spaces the pseudo inner product is a true inner product.

Pseudo Symmetry and Reciprocity

For a reciprocal medium the Lorentz reciprocity theorem (1.20) provides a form that is very useful for a variety of applications and exhibits pseudo symmetry (symmetry under a pseudo inner product) of field operators. For instance, consider in the vicinity of the origin an inhomogeneous, anisotropic medium characterized by symmetric dyadics, $\underline{\varepsilon} = \underline{\varepsilon}^T, \underline{\mu} = \underline{\mu}^T$ (reciprocal medium). We assume the space is unbounded, and as $r \rightarrow \infty$ the medium becomes free space. Let $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ be the fields caused by sources $\mathbf{J}_{e1}, \mathbf{J}_{m1}$ and $\mathbf{J}_{e2}, \mathbf{J}_{m2}$, respectively. Then, (1.20) becomes

$$\langle \mathbf{E}_2 \cdot \mathbf{J}_{e1} \rangle_p - \langle \mathbf{H}_2 \cdot \mathbf{J}_{m1} \rangle_p = \langle \mathbf{E}_1 \cdot \mathbf{J}_{e2} \rangle_p - \langle \mathbf{H}_1 \cdot \mathbf{J}_{m2} \rangle_p. \quad (3.10)$$

We can think of $\mathbf{E}$ ($\mathbf{H}$) in terms of a linear operator $\mathbf{L}_E$ ($\mathbf{L}_H$) that produces the electric (magnetic) field from current sources¹⁸ $\mathbf{J}_e, \mathbf{J}_m$. Let

$$\Psi_i = \begin{bmatrix} \mathbf{J}_{ei} \\ \mathbf{J}_{mi} \end{bmatrix}, \quad F_i = \begin{bmatrix} \mathbf{E}_i \\ -\mathbf{H}_i \end{bmatrix},$$

and consider the operator¹⁹

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_E \\ -\mathbf{L}_H \end{bmatrix} = \begin{bmatrix} L_{e,e} & L_{e,m} \\ -L_{m,e} & -L_{m,m} \end{bmatrix}$$

¹⁷Indeed, Hermitian matrices (the matrix analog of self-adjointness) have very special properties (such as real eigenvalues, basis of eigenvectors, etc.), whereas complex-valued matrices symmetric about the main diagonal, termed *complex-symmetric matrices* (the matrix analog of pseudo self-adjointness), do not possess especially useful properties.

¹⁸For instance, (1.86) provides a form for $L_{E(H)}$ in the special case of isotropic, homogeneous media.

¹⁹Using this notation, application of the operators follows the usual rules of matrix multiplication.

such that $\mathbf{L}\Psi_i = \mathbf{F}_i$, or

$$\begin{bmatrix} L_{e,e} & L_{e,m} \\ -L_{m,e} & -L_{m,m} \end{bmatrix} \begin{bmatrix} \mathbf{J}_{ei} \\ \mathbf{J}_{mi} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_i \\ -\mathbf{H}_i \end{bmatrix}.$$

Then from (3.10) we have

$$\langle \mathbf{L}\Psi_2, \Psi_1 \rangle_p = \langle \Psi_2, \mathbf{L}\Psi_1 \rangle_p$$

such that the operator is pseudo-symmetric. In this case it is obvious that symmetry under a pseudo inner product is simply a statement of reciprocity. As mentioned previously, for static fields the pseudo inner product is a legitimate inner product since we deal with real-valued quantities. In this case the operator $\mathbf{L}$ is symmetric (self-adjoint if the domains are appropriately chosen).

If desired, the two independent Maxwell curl equations from (1.9) can be written in operator form as

$$\mathbf{L}_M \mathbf{F} = \begin{pmatrix} -i\omega \underline{\epsilon} \cdot & -\nabla \times \underline{\mathbf{I}} \cdot \\ -\nabla \times \underline{\mathbf{I}} \cdot & i\omega \underline{\mu} \cdot \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ -\mathbf{H} \end{pmatrix} = \Psi = \begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix},$$

which is useful for formal manipulations. Comparing with $\mathbf{L}\Psi = \mathbf{F}$, the operator $\mathbf{L}_M$ is obviously the formal inverse to $\mathbf{L}$, i.e., $\mathbf{L}\mathbf{L}_M = \mathbf{L}_M\mathbf{L} = I$; therefore, $\mathbf{L}^{-1} = \mathbf{L}_M$. If reciprocity holds, then $\mathbf{L}_M$ is pseudo-symmetric.

3.5 Definiteness and Convergence in Energy

3.5.1 General Concepts

Since for self-adjoint or merely symmetric operators $\langle Ax, x \rangle \in \mathbf{R}$, certain further distinctions may be made.

Definition 3.19. *A self-adjoint or symmetric operator $A : H \rightarrow H$ is said to be nonnegative if*

$$\langle Ax, x \rangle \geq 0$$

for all $x \in H$, denoted as $A \geq 0$. An operator is said to be (strictly) positive if $\langle Ax, x \rangle > 0$ for all $x \neq 0$ in H , denoted as $A > 0$. An operator is said to be positive definite if there is a constant $k > 0$ such that

$$\langle Ax, x \rangle \geq k \|x\|^2$$

for all $x \in H$.

Nonpositive, negative, and negative definite operators can be similarly defined, and in general we will call an operator *definite* if $\langle Ax, x \rangle \gtrless 0$ (i.e., a strict inequality exists). It is clear that positive definite operators are

also positive, but the converse is not necessarily true. A symmetric or self-adjoint operator that doesn't fit into one of these categories is called *indefinite*. In this case $\langle Ax, x \rangle$ will be positive for some $x \in H$ and negative for other $x \in H$.

An operator A satisfying the definition for positive-definiteness is also said to be *bounded below*. An operator A is said to be *bounded above* if there is a constant K such that $\langle Ax, x \rangle \leq K \|x\|^2$ for all $x \in H$. An operator on a Hilbert space is bounded if and only if it is bounded above and below [12, p. 364]. We also have the following theorem.

Theorem 3.14. *Let $A : H \rightarrow H$ be a bounded linear operator. Then, the operators A^*A and AA^* are nonnegative. If $Ax = 0$ has only the trivial solution $x = 0$, then A^*A is positive.*

Proof. $\langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0$, $\langle AA^*x, x \rangle = \langle A^*x, A^*x \rangle = \|A^*x\|^2 \geq 0$. Clearly, $\|Ax\|^2 > 0$ unless $Ax = 0$, but this would imply $x = 0$. ■

Note that the product of two commuting positive operators is positive, but the product of two positive operators is not necessarily a positive operator. Furthermore, a *square root* of a positive operator A is a self-adjoint operator B such that $B^2 = A$.

If an operator is not symmetric, then generally $\langle Ax, x \rangle \in \mathbb{C}$. In this case there is a concept related to definiteness.

Definition 3.20. *An operator $A : H \rightarrow H$ is dissipative if, for all $x \in D_A$,*

$$\operatorname{Im} \langle Ax, x \rangle \geq 0. \quad (3.11)$$

If A is bounded, then (3.11) is equivalent to the condition that its imaginary component $(A - A^*)/2i$ (See Theorem 3.9) be nonnegative.

In accordance with the desire to make Hilbert spaces analogous to geometrical spaces, it is worthwhile to note an analogy between operators on a Hilbert space and numbers in the complex plane. Under this analogy the adjoint plays the role of complex conjugation, self-adjoint operators are analogous to real numbers ($A = A^*$), and unitary operators are analogous to complex numbers having unit magnitude ($AA^* = I$). Also, positive (negative) operators are analogous to positive (negative) numbers.

Definiteness and Fields

Considering electric currents and using the operator $L_{e,e}$, which produces the electric field from the electric current, it is clear that

$$\langle L_{e,e} \mathbf{J}_e, \mathbf{J}_e \rangle = \langle \mathbf{E}, \mathbf{J}_e \rangle = \int_{\Omega} \mathbf{E}(\mathbf{r}) \cdot \bar{\mathbf{J}}_e(\mathbf{r}) d\Omega$$

has units of *watts* and represents a complex power density. In general $L_{e,e}$ is pseudo-symmetric, and so $\langle L_{e,e} \mathbf{J}_e, \mathbf{J}_e \rangle$ is generally not even real-valued, let alone strictly positive or negative. However, for static fields $L_{e,e}$ is symmetric and $\langle L_{e,e} \mathbf{J}_e, \mathbf{J}_e \rangle$ will be definite under fairly general conditions. For instance, for applied currents $\mathbf{J}_e = \mathbf{J}_e^i$, $\langle L_{e,e} \mathbf{J}_e, \mathbf{J}_e \rangle < 0$ for a source supplying power. For induced currents in a passive medium, $\mathbf{J}_e = \underline{\sigma}_e \cdot \mathbf{E}$ and $\langle L_{e,e} \mathbf{J}_e, \mathbf{J}_e \rangle > 0$, representing dissipated power.

3.5.2 Examples Relating to Definiteness

1. The identity operator $I : H \rightarrow H$ defined by $Ix \equiv x$ is a bounded, self-adjoint positive operator since $\langle Ix, x \rangle = \langle x, x \rangle > 0$ for all $x \neq 0$ in H .
2. The multiplication operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv f(t)x(t),$$

where $f(t)$ is a bounded real-valued nonnegative (positive) function, is a bounded, self-adjoint nonnegative (positive) operator. Indeed,

$$\langle Ax, x \rangle = \int_a^b f(t)x(t)\overline{x(t)} dt = \int_a^b f(t)|x(t)|^2 dt \geq 0$$

if $f(t) \geq 0$ on $[a, b]$.

3. The operator $A : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ with matrix representation $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is indefinite, as can be seen by examining $\langle Ax, x \rangle$ with $x = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ ($\langle Ax, x \rangle = -2 < 0$) and $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ ($\langle Ax, x \rangle = 6 > 0$).
4. The differentiation operator $A : \mathbf{L}^2(0, b) \rightarrow \mathbf{L}^2(0, b)$ (with $0 < b < \infty$) defined by

$$(Ax)(t) \equiv -\frac{d^2}{dt^2}x(t) = -x''(t),$$

$$D_A \equiv \{x : x(t), x'(t), x''(t) \in \mathbf{L}^2(0, b), x(0) = x(b) = 0\}$$

is an unbounded, self-adjoint, positive operator. To see this, consider

$$\begin{aligned} \langle Ax, x \rangle &= -\int_0^b x''(t)\overline{x(t)} dt \\ &= -x'(t)\overline{x(t)} \Big|_0^b + \int_0^b x'(t)\overline{x'(t)} dt = \int_0^b |x'(t)|^2 dt \geq 0, \end{aligned}$$

showing that A is at least nonnegative. Because $x(0) = 0$,

$$|x(t)|^2 = \left| \int_0^t x'(\tau) d\tau \right|^2 \leq \int_0^t 1^2 d\tau \int_0^t |x'(\tau)|^2 d\tau \leq t \int_0^b |x'(\tau)|^2 d\tau.$$

Therefore,

$$\int_0^b |x'(\tau)|^2 d\tau = \langle Ax, x \rangle \geq \frac{|x(t)|^2}{t} > 0,$$

and so $\langle Ax, x \rangle > 0$.

5. The negative Laplacian operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv -\nabla^2 x(\mathbf{t}),$$

$$D_A \equiv \{x : x(\mathbf{t}), \nabla^2 x(\mathbf{t}) \in \mathbf{L}^2(\Omega), x(\mathbf{t})|_S = 0\}$$

is an unbounded, self-adjoint, nonnegative operator. Indeed, from the first scalar Green's theorem

$$\begin{aligned} \langle Ax, x \rangle &= - \int_{\Omega} \bar{x} \nabla^2 x d\Omega \\ &= - \oint_S \bar{x} \frac{\partial x}{\partial n} dS + \int_{\Omega} \overline{\nabla x} \cdot \nabla x d\Omega = \int_{\Omega} |\nabla x|^2 d\Omega \geq 0. \end{aligned}$$

6. The vector differential operator $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)^3$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \nabla \times \nabla \times \mathbf{x}(\mathbf{t}),$$

$$D_A \equiv \left\{ \mathbf{x} : \mathbf{x}(\mathbf{t}), \nabla \times \mathbf{x}(\mathbf{t}), \nabla \times \nabla \times \mathbf{x}(\mathbf{t}) \in \mathbf{L}^2(\Omega)^3, \right.$$

$$\left. \mathbf{n} \times \mathbf{x}(\mathbf{t})|_S = 0 \right\}$$

is an unbounded, self-adjoint, nonnegative operator. To see that the operator A is nonnegative, consider the first vector Green's theorem,

$$\begin{aligned} \langle A\mathbf{x}, \mathbf{x} \rangle &= \int_{\Omega} \bar{\mathbf{x}} \cdot (\nabla \times \nabla \times \mathbf{x}) d\Omega \\ &= - \oint_S (\mathbf{n} \times \bar{\mathbf{x}}) \cdot (\nabla \times \mathbf{x}) dS + \int_{\Omega} \overline{(\nabla \times \mathbf{x})} \cdot (\nabla \times \mathbf{x}) d\Omega \\ &= \int_{\Omega} |\nabla \times \mathbf{x}|^2 d\Omega \geq 0. \end{aligned}$$

3.5.3 Convergence in Energy

We previously introduced the notion of strong and weak convergence for inner product spaces (see Section 2.5.3). Recall that a sequence of elements $x_1, x_2, \dots$ in an inner product space S is said to be strongly convergent to an element $x \in S$ (denoted $x_n \rightarrow x$) if $\|x_n - x\| \rightarrow 0$, and weakly convergent to an element $x \in S$ (denoted as $x_n \xrightarrow{w} x$) if $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ for all $y \in S$. For positive operators (and positive definite operators as a special case) we can introduce another measure of convergence by using a special inner product and norm called the *energy inner product* and *energy norm* [4, p. 91].

Definition 3.21. Consider a positive operator $A : H \rightarrow H$. The energy inner product is a mapping $H \times H \rightarrow \mathbb{C}$ that associates, for a given A , an ordered pair $x, y \in D_A$ with a scalar denoted by

$$\langle x, y \rangle_e \equiv \langle Ax, y \rangle \quad (3.12)$$

obeying the usual properties of an inner product.

Since the energy inner product obeys the usual inner product rules, it induces a norm, called the energy norm. Denoting the energy norm as $\|x\|_e$, we obtain

$$\|x\|_e = \sqrt{\langle x, x \rangle_e} = \sqrt{\langle Ax, x \rangle}. \quad (3.13)$$

We define the Hilbert space H_e , associated with a particular positive operator A , as an energy inner product space complete in the norm $\|x\|_e$. Within this space we have the concept of an orthonormal set,

$$\langle x_n, x_m \rangle_e = \delta_{nm}. \quad (3.14)$$

An orthonormal set $\{x_n\}$ is a basis of H_e if each $y \in H_e$ can be written in a unique way as

$$y = \sum_{n=1}^{\infty} \langle y, x_n \rangle_e x_n.$$

Because $\|x\|_e^2 = \langle Ax, x \rangle \geq k \|x\|^2$ for positive definite operators, it is easy to see that

$$\|x\| \leq \frac{1}{\sqrt{k}} \|x\|_e.$$

Now we are ready to define the concept of *convergence in energy*.

Definition 3.22. A sequence of elements $x_1, x_2, \dots$ in a Hilbert space H is said to converge in energy to an element $x \in H$ (denoted $x_n \xrightarrow{e} x$) if $\|x_n - x\|_e \rightarrow 0$.

We have the following interrelationships among convergence.

Theorem 3.15. Assume $\|Ax_n\|$ is bounded. Then

- strong convergence implies convergence in energy,
- convergence in energy implies $Ax_n \xrightarrow{w} Ax$,
- if A is positive definite, convergence in energy implies strong convergence.

Proof. (a) $\|x_n - x\|_e^2 = |\langle A(x_n - x), (x_n - x) \rangle| \leq \|A(x_n - x)\| \|x_n - x\| = \|Ax_n - Ax\| \|x_n - x\| \leq (\|Ax_n\| + \|Ax\|) \|x_n - x\|$. Since $\|x_n - x\| \rightarrow 0$ by strong convergence, and $\|Ax_n\|$ is bounded, then $\|x_n - x\|_e \rightarrow 0$.

(b) $\|x_n - x\|_e^2 \rightarrow 0$ from (a) implies $|\langle A(x_n - x), (x_n - x) \rangle| \rightarrow 0$ for $x \in H$, which leads to $Ax_n \xrightarrow{w} Ax$ as $x_n \xrightarrow{w} x$.

(c) Because $\|x_n - x\| \leq (1/\sqrt{k}) \|x_n - x\|_e$, (c) is proved. ■

3.6 Compact Operators

3.6.1 General Concepts

Compact (completely continuous) operators comprise an important subset of bounded operators, especially within the theory of integral equations. Compact operators are operators that have a finite-dimensional, or an almost finite-dimensional, range. In infinite-dimensional spaces compact operators have the simplest properties of all infinite-dimensional operators. This is especially true when one considers the spectral theory covered in the next chapter, where it is shown that the basic elements of spectral theory for compact self-adjoint operators are similar to those from the spectral theory of matrix operators. Also note that the theory of linear compact operators is a generalization of Fredholm's theory of integral equations. In particular, compactness is a very important concept in formulating boundary value problems of applied electromagnetics leading to Fredholm integral equations. The solution of operator equations involving compact operators is discussed in a later section.

Definition 3.23. *A bounded linear operator $A : H_1 \rightarrow H_2$ is compact²⁰ if, for each bounded sequence $\{x_n\} \subset H_1$, there is a subsequence $\{x_{n_i}\}$ such that $\{Ax_{n_i}\}$ converges in H_2 .*

Note that every compact linear operator is bounded, but the converse is not necessarily true (for example, consider the identity operator). It can be shown that every linear bounded operator with finite-dimensional range²¹ is compact. Therefore, necessarily, $A : \mathbf{C}^n \rightarrow \mathbf{C}^m$ is compact (e.g., matrix operators are compact).

Information on the range of a compact operator is given in the following.

Theorem 3.16. *Let $A : H_1 \rightarrow H_2$ be a compact linear operator. Given any $\varepsilon > 0$, there exists a finite-dimensional subspace M of R_A such that*

$$\inf\{\|Ax - m\|\} \leq \varepsilon \|x\|$$

for $m \in M, x \in H_1$.

Therefore, the finite-dimensional subspace M is very close to (in a sense within ε of the possibly infinite-dimensional) R_A .

The next theorem provides some interesting properties of the important operator $I - A$ for A compact; these properties are useful for considering solutions of second-kind operator equations.

Theorem 3.17. *Let $T = I - A$, with I the identity operator on a Hilbert space H , and $A : H \rightarrow H$ a compact operator. Then*

²⁰A frequently used alternate definition is that an operator A is compact if it maps any bounded set of H_1 into a compact set (i.e., one having compact closure) of H_2 .

²¹A linear operator with a finite-dimensional range is also called a finite-rank operator.

- a. N_T, N_{T^*} are finite-dimensional and have the same dimension.
- b. $H - R_{T^*} = N_T, H - R_T = N_{T^*}$.
- c. $R_T = H$ if and only if $N_T = \{0\}$, and $R_{T^*} = H$ if and only if $N_{T^*} = \{0\}$.
- d. R_T, R_{T^*} are closed subspaces of H .

Thus, the operator $I - A$ for A compact has particularly nice geometric properties.

We will state without proof some properties of compact operators.

- Let $A, B : H_1 \rightarrow H_2$ be compact linear operators and $C : H_3 \rightarrow H_1, D : H_2 \rightarrow H_3$ be bounded operators. Then $A + B, AC$, and DA are compact linear operators.

In practical applications it is sometimes convenient to represent an operator as a product of operators. Compactness is then shown for the product operator by simply showing that one operator is bounded and the other is compact. Also, the fact that $A + B$ is compact is often used in the theory of integral equations (for example, see [14, p. 346]). In general, any linear combination of compact linear operators is also a compact operator.

- An operator $A : H_1 \rightarrow H_2$ is compact if and only if its adjoint $A^* : H_2 \rightarrow H_1$ is compact. If AA^* is compact, then A is also compact.
- Let $\{A_n\} : H_1 \rightarrow H_2$ be a sequence of compact operators. If

$$\|A_n - A\| \rightarrow 0$$

for some $A : H_1 \rightarrow H_2$, then A is compact (this important property is often used to prove compactness).

The above property provides insight into the fact that compact operators either have finite rank or are nearly finite rank, since the limit of a sequence of finite-rank (and hence compact) operators is compact. Furthermore, any compact operator in a Hilbert space can be approximated in norm arbitrarily closely by a sequence of operators of finite rank [15, p. 507]. In fact, on a Hilbert space, and most Banach spaces [16, pp. 183–184], a compact operator can be expressed as the sum of a finite-rank (degenerate) operator and an operator with a small norm.

The next property is also useful for proving compactness [12, p. 358].

- If A is a compact operator in H , and $\{x_n\}$ is an orthonormal sequence in H , then $\lim_{n \rightarrow \infty} Ax_n = 0$.

It is also seen that compact operators are very well behaved when applied in the “forward direction.”

- An operator A is compact if and only if $x_n \xrightarrow{w} x$ implies $Ax_n \rightarrow Ax$.

Recall that this means that if A is compact, weak convergence, i.e., $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ for all $y \in H$, implies strong convergence, $\|Ax_n - Ax\| \rightarrow 0$. As an example, for a sequence of functions $\{x_n(t)\}$ in $L^2(a, b)$ with the usual inner product, weak convergence

$$\int_a^b (x_n(t) - x(t)) \overline{y(t)} dt \rightarrow 0$$

for all $y \in H$ implies convergence in the norm²²

$$\left(\int_a^b |Ax_n(t) - Ax(t)|^2 dt \right)^{1/2} \rightarrow 0.$$

Thus, while bounded linear operators map strongly convergent sequences into strongly convergent sequences (and for operators boundedness is equivalent to continuity), compact linear operators map weakly convergent sequences into strongly convergent sequences. Compact operators can then be thought of as, in a sense, strongly bounded operators, or equivalently, strongly continuous operators—hence the name *completely continuous operators*. However, the inverse of a compact operator is unbounded, as shown in Theorem 3.37, and this fact presents some difficulties in applications.

Finally, we have the following.

- An operator A is compact if for any sequences x_n and y_n weakly converging to x and y ($x_n \xrightarrow{w} x$, $y_n \xrightarrow{w} y$) the inner product $\langle Ax, y \rangle$ is a weakly continuous function of x and y such that

$$\lim_{n \rightarrow \infty} \langle Ax_n, y_n \rangle = \langle Ax, y \rangle.$$

3.6.2 Examples Relating to Compactness

1. The identity operator ($Ix = x$) on an infinite-dimensional space is bounded but not compact (I is compact only on a finite-dimensional space), but the zero operator ($0x = 0$) is compact. The fact that the identity operator I is not compact is relevant to the theory of Fredholm operators represented in the form $I - A$, where A is a compact operator, leading to the existence of the bounded inverse operator $(I - A)^{-1}$.

²²As a concrete example, on $H = L^2(0, \pi)$ the sequence $\{x_n\} = \{\sin(nt)\}$ converges weakly to 0, i.e., $\int_0^\pi (\sin(nt) - 0) \overline{y(t)} dt \rightarrow 0$ for $y \in H$ (try $y = 1$ or $y = t$ as a simple test). The sequence does not converge in norm to 0, since $(\int_0^\pi (\sin^2(nt) - 0) dt)^{1/2} = \sqrt{\pi/2}$. However, subsequent to application of the compact operator $(Ax)(t) \equiv \int_0^t x(s) ds$ we have $\{Ax_n\} = \{\frac{1 - \cos tn}{n}\}$, which converges strongly to 0, i.e., $\left(\int_a^b \left| -\frac{\cos tn - 1}{n} - 0 \right|^2 dt \right)^{1/2} \rightarrow 0$.

2. The operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ corresponding to the infinite matrix $[a_{ij}]$, where $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$, is bounded and compact²³ in $\mathbf{l}^2$. The proof is based on the previously stated criterion that a sequence of finite-rank (compact) operators A_n is convergent in the $\mathbf{l}^2$ norm to the operator A such that $\|A_n - A\| \rightarrow 0$ [17, p. 86]. As a special case, the operator corresponding to an infinite diagonal matrix $\text{diag}[\lambda_1, \lambda_2, \lambda_3, \dots]$ for which $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ is compact (and self-adjoint if all λ 's are real-valued).

3. The multiplication operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv f(\mathbf{t})x(\mathbf{t})$$

is a bounded linear operator (assuming $k = \max_{\mathbf{t} \in \Omega} |f(\mathbf{t})| < \infty$) that is not compact (unless $f \equiv 0$) [17, pp. 87–88].

4. The integral operator $A : \mathbf{L}^2(\Omega)^m \rightarrow \mathbf{L}^2(\Omega)^m$ defined by

$$(A\mathbf{x})(\mathbf{t}) \equiv \int_{\Omega} \underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \cdot \mathbf{x}(\mathbf{s}) d\Omega,$$

where $\Omega \subset \mathbf{R}^n$ with $\underline{\mathbf{k}}(\mathbf{t}, \mathbf{s}) \in \mathbf{L}^2(\Omega \times \Omega)^{m \times m}$ (in the sense that each component of the $(m \times m)$ -tuple dyadic kernel $\underline{\mathbf{k}}(\mathbf{t}, \mathbf{s})$ belongs to $\mathbf{L}^2(\Omega \times \Omega)$), is bounded, and it is compact if and only if each component of the integral operator $(A\mathbf{x})(\mathbf{t})$ is compact in $\mathbf{L}^2(\Omega)$.

As a special case, the integral operator $A : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ defined by

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} k(\mathbf{t}, \mathbf{s})x(\mathbf{s}) d\Omega$$

with $k(\mathbf{t}, \mathbf{s}) \in \mathbf{L}^2(\Omega \times \Omega)$ is bounded (Hilbert–Schmidt kernel, $\|A\| \leq \|k\|$) and compact.²⁴

A Hilbert–Schmidt kernel $k(\mathbf{t}, \mathbf{s})$ may itself be (and often is) unbounded as a function of $\mathbf{t}$ and $\mathbf{s}$, yet it always generates a compact operator (more generally, all Hilbert–Schmidt operators are compact). Also, the conditions for a kernel to be Hilbert–Schmidt are sufficient, but not necessary, for an integral operator to be compact.

This example can be also generalized to show that a bounded in $\mathbf{L}^p(\Omega)$ ($1 < p < \infty$) integral operator $A : \mathbf{L}^p(\Omega) \rightarrow \mathbf{L}^p(\Omega)$ with $\|A\| \leq \|k\|$ and $\|k\|^p = \int_{\Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})|^p d\Omega_s d\Omega_t < \infty$ is compact.

²³The condition $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$ for compactness of matrix operators from $\mathbf{l}^2$ into $\mathbf{l}^2$ is often used in applications. However, this condition is a very strong one, and it is not necessary that every compact operator from $\mathbf{l}^2$ into $\mathbf{l}^2$ satisfy this condition.

²⁴The proof of compactness is based on the approximation of $\|k\|$ by a sequence of continuous degenerate kernels $k_n(t, s)$ such that $\|k - k_n\| \rightarrow 0$ and the fact that finite-rank operators A_n are compact. Using $\|A\| \leq \|k\|$ for a bounded operator, $\|A - A_n\| \leq \|k - k_n\| \rightarrow 0$, then $A_n \rightarrow A$ and A is compact (for a complete proof, see [16, p. 183] and [4, pp. 155–158]).

5. The integration operator $A : L^2(a, b) \rightarrow L^2(a, b)$ defined by $(Ax)(t) \equiv \int_a^t x(s) ds$ is compact.
6. The weakly singular integral operator defined by

$$(Af)(s) \equiv \int_{\Omega} \frac{k(t, s)}{|t - s|^m} f(t) d\Omega,$$

where $\Omega \subset \mathbf{R}^n$, $0 \leq m < n$, and $k(t, s)$ is bounded on $\Omega \times \Omega$, is compact in $L^2(\Omega)$ [4, p. 160]. This also holds for integrals of the form

$$(Af)(s) = \int_{\Gamma} \frac{k(t, s)}{|t - s|^m} f(t) d\Gamma$$

where Γ is a smooth n -dimensional surface in an $(n + 1)$ dimensional space.

Therefore, the usual free-space Green's function kernel

$$g(t, s) = \frac{e^{-ik|t-s|}}{4\pi|t-s|}$$

in a bounded region $\Omega \subset \mathbf{R}^3$ generates a compact (nonself-adjoint) operator

$$(Af)(s) = \int_{\Omega} g(t, s) f(t) d\Omega$$

in $L^2(\Omega)$. Similarly, in two dimensions the Green's function

$$g(t, s) = \frac{1}{4i} H_0^{(2)}(k|t-s|)$$

generates a weakly singular ($H_0^{(2)}(x) \sim \ln(x/2)$ as $x \rightarrow 0$) compact integral operator in $L^2(\Omega)$, $\Omega \subset \mathbf{R}^2$. However, if the kernel has the form of sufficiently many derivatives of g , then the resulting operator will not be compact.

Note if $0 < m < n/2$, then the operator with a weak singularity is a Hilbert-Schmidt operator in $L^2(\Omega)$ such that

$$\int_{\Omega} \int_{\Omega} \frac{d\Omega_t d\Omega_s}{|t - s|^{2m}} < \infty.$$

Also note that the one-dimensional Cauchy singular operator

$$(Af)(s) = \int_{\Gamma} \frac{f(t)}{|t - s|} dt$$

is bounded in $L^p(\Gamma)$ for $1 < p < \infty$ but not compact in $L^p(\Gamma)$. However, the operator

$$A(gf) - gA(f) = \int_{\Gamma} \frac{g(t) - g(s)}{|t - s|} f(t) dt$$

is compact in $\mathbf{L}^p(\Gamma)$, where $g(t)$ is a continuous function on Γ (Γ is a sufficiently smooth (Lyapunov) curve). This is an important result in the theory of one-dimensional singular operators.

7. The operator $A : \mathbf{L}^2(\Omega)^3 \rightarrow \mathbf{L}^2(\Omega)^3$ defined by

$$(A\mathbf{J})(\mathbf{r}) \equiv \int_s \mathbf{n} \times \left[\nabla \frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \times \mathbf{J}(\mathbf{r}') \right] dS',$$

which arises from the magnetic field integral equation (1.118), is compact [14, pp. 346–348].

8. Consider the integral operator $A : \mathbf{L}^2(-a, a) \rightarrow \mathbf{L}^2(-a, a)$ defined by

$$(Ax)(t) \equiv \int_{-a}^a x(s)k(t, s)w(s) ds$$

with the kernel $k(t, s) = \ln|t-s|$ and with the weight function $w(s) = 1/\sqrt{a^2-s^2}$. The operator

$$(Ax)(t) = \int_{-a}^a x(s) \frac{\ln|t-s|}{\sqrt{a^2-s^2}} ds$$

is compact in $\mathbf{L}^2(-a, a)$ and, moreover, $\ln|t-s|$ is a Hilbert–Schmidt kernel with respect to the weight function w such that

$$\int_{-a}^a \int_{-a}^a \frac{|\ln|t-s||^2}{\sqrt{a^2-s^2}\sqrt{a^2-t^2}} ds dt < \infty.$$

9. The integral operator $A : \mathbf{C}(\Omega) \rightarrow \mathbf{C}(\Omega)$, defined by

$$(Ax)(\mathbf{t}) \equiv \int_{\Omega} k(\mathbf{t}, \mathbf{s})x(\mathbf{s}) d\mathbf{s},$$

where $\Omega \subset \mathbf{R}^n$ and $k(\mathbf{t}, \mathbf{s}) \in \mathbf{C}(\Omega \times \Omega)$, is bounded with

$$\|A\| = \max_{\mathbf{t} \in \Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\mathbf{s}$$

assuming $\|k\| = \max_{\mathbf{t} \in \Omega} \int_{\Omega} |k(\mathbf{t}, \mathbf{s})| d\mathbf{s} < \infty$. Also, suppose that there are constants c and $m < n$ such that

$$|k(\mathbf{t}, \mathbf{s})| \leq c|\mathbf{t}-\mathbf{s}|^{-m}$$

for $\mathbf{t}, \mathbf{s} \in \Omega$ and $\mathbf{t} \neq \mathbf{s}$. Then, A is compact in $\mathbf{C}(\Omega)$. The proof is based on the Arzela criterion for a compact set (see Section 2.2.2; also [16,

pp. 181–182]).²⁵ If Ω is not bounded, then $A : \mathbf{C}(\Omega) \rightarrow \mathbf{C}(\Omega)$ is a bounded operator that is not compact. This is important in the analysis of Fourier integrals and convolution-type operators with the kernel $k(\mathbf{t}, \mathbf{s}) = k(\mathbf{t} - \mathbf{s})$.

10. Consider the operator $A : H^{(n)} \rightarrow H^{(n)}$ given by a matrix of operators $A_{ij} : H \rightarrow H$ such that

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix},$$

where $H^{(n)} = H \oplus H \oplus \cdots \oplus H$ (n times) and H may be infinite-dimensional. The operator $A : H^{(n)} \rightarrow H^{(n)}$ is compact if and only if each operator $A_{ij} : H \rightarrow H$ is compact. Many problems of applied electromagnetics lead to a coupled system of integral equations where an operator can be represented by a matrix of individual integral operators. Usually, if diagonal operators A_{ii} are compact, then off-diagonal operators A_{ij} ($i \neq j$) will be compact as well.

3.7 Continuity and Compactness of Matrix Operators

In this section we present some criteria governing properties of continuity and compactness of matrix operators defined on sequence spaces. In particular, this becomes useful when a problem is reduced from a function space to a matrix level. There are important theorems for matrix operators that can be directly applied to electromagnetic problems to justify existence and uniqueness of solutions by means of properties of matrix operators. In addition, it is often easier to work with matrix operators than with the corresponding operators on the level of a function space.

²⁵Basically, to show that the integral operator A is compact in $\mathbf{C}(a, b)$ it is sufficient to prove that the following conditions are satisfied for its kernel $k(t, s)$:

$$\begin{aligned} \max_{a \leq t \leq b} \int_a^b |k(t, s)| ds &< \infty, \\ \lim_{\delta \rightarrow 0} \int_a^b |k(t + \delta, s) - k(t, s)| ds &= 0, \quad a \leq t \leq b. \end{aligned}$$

3.7.1 Matrix Operators in l^2

A sufficient condition for compactness of $A : l^2 \rightarrow l^2$ corresponding to the infinite matrix $[a_{ij}]$ is discussed in Section 3.6.2. Example 2.

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty. \quad (3.15)$$

which is a strong criterion in the sense that not every compact operator will satisfy this condition. In general, operators that meet the requirement (3.15) are Hilbert–Schmidt- and Fredholm-type operators with finite absolute norm.

For the special case

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < 1. \quad (3.16)$$

there is a theorem related to solvability of matrix equations of the second kind, $(I - A)x = y$ [17].

Theorem 3.18. *The infinite system of equations*

$$x_i - \sum_{j=1}^{\infty} a_{ij} x_j = y_i,$$

$i = 1, 2, \dots$, where elements of the matrix $[a_{ij}]$ are such that (3.16) is satisfied, has a unique solution $\xi = (\xi_1, \xi_2, \dots)$ in l^2 for every $y = (y_1, y_2, \dots) \in l^2$. The truncated system of equations

$$x_i - \sum_{j=1}^n a_{ij} x_j = y_i,$$

$i = 1, 2, \dots, n$, has a unique solution $(x_1^{(n)}, \dots, x_n^{(n)})$ and, furthermore, $(x_1^{(n)}, \dots, x_n^{(n)}, 0, 0, \dots)$ converges to ξ in l^2 as $n \rightarrow \infty$.

In problems of applied electromagnetics where the solution is restricted by a finite energy condition, such that the sequence space $\tilde{l}^2$ is encountered for the coefficients in an eigenfunction expansion (see Example 2 in Section 2.1.3), a sufficient condition for compactness of a matrix operator $A : \tilde{l}^2 \rightarrow \tilde{l}^2$ can be written as [18], [19]

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 \frac{|i|}{|j|} < \infty.$$

Other conditions for continuity and compactness are discussed in [20]. For example, consider a matrix operator $A : l^2 \rightarrow l^2$ corresponding to an

infinite matrix of elements $[a_{ij}]$. Then

$$\|Ax\|^2 = \sum_{i=1}^{\infty} a_{ij} x_j \bar{a}_{ik} \bar{x}_k \leq \sup_{1 \leq k < \infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij} a_{ik}| \|x\|^2 = c \|x\|^2,$$

where $c = \sup_{1 \leq k < \infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij} a_{ik}|$. The operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ is bounded (continuous) if $\|A\| \leq c$, which results in the condition

$$\|A\| \leq \left(\sup_{1 \leq k < \infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij} a_{ik}| \right)^{1/2} < \infty. \quad (3.17)$$

The matrix operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ is completely continuous (compact) if, in addition to (3.17), we also have

$$\|A\| = \lim_{s \rightarrow \infty} \sup_{1 \leq k < \infty} \sum_{j=1}^{\infty} \sum_{i=s}^{\infty} |a_{ij} a_{ik}| = 0. \quad (3.18)$$

The following lemma gives a criterion for boundedness of $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ [20].

Lemma 3.19. (Schur) Assume that $[a_{ij}] = [\bar{a}_{ji}]$ and $\sup_i \sum_{j=1}^{\infty} |a_{ij}| \leq c$. Then the matrix operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ is bounded, such that $\|A\| \leq c$.

Proof. For a symmetric operator $Ax = A^*x$ it can be shown that

$$|\langle Ax, x \rangle| = \left| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} x_j \bar{x}_i \right| \leq \sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}| \|x\|^2 \leq c \|x\|^2.$$

Therefore, $\|A\| \leq c$. ■

The criteria for continuity and compactness of matrix operators in $\mathbf{l}^2$ discussed here have been used in the analysis of various electromagnetic problems leading to an infinite system of linear algebraic equations of the second kind. See, for example, [21]–[29], among others.

In the next section we summarize some useful theorems for continuity and compactness of matrix operators acting in different sequence spaces. Some applications of these theorems are shown in Section 9.5.

3.7.2 Gribanov's Theorems

According to [30], we can define a class of continuous matrix operators $S(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$ that continuously map the sequence space $\mathbf{l}^1$ into $\mathbf{l}^2$. Within $S(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$ we define a class of completely continuous (compact) operators $V(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$, such that $V(\mathbf{l}^1 \rightarrow \mathbf{l}^2) \subset S(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$. For a matrix operator A

corresponding to an infinite matrix with elements $[a_{ij}]$, we define a matrix operator A_{ii} as

$$A_{ii} \equiv \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1i} & 0 & 0 & \dots \\ a_{21} & a_{22} & \dots & a_{2i} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ii} & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Definition 3.24. A matrix operator $A \in S(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$ is called ω -continuous if

$$\lim_{i \rightarrow \infty} \|A - A_{ii}\| = 0.$$

The class of ω -continuous operators is denoted by $V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^2)$, and $V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^2) \subseteq V(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$.

Classes of continuous $S(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$, ω -continuous $V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^2)$, and completely continuous $V(\mathbf{l}^1 \rightarrow \mathbf{l}^2)$ matrix operators can be generalized to $S(\mathbf{l}^1 \rightarrow \mathbf{l}^p)$, $V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^p)$, and $V(\mathbf{l}^1 \rightarrow \mathbf{l}^p)$, $p \geq 1$, respectively, and other sequence spaces used in the theorems to follow. Below we summarize some of the theorems presented in [30] for continuity and ω -continuity of matrix operators acting in certain sequence spaces.²⁶

Theorem 3.20. If a matrix operator A is ω -continuous from $\mathbf{l}^1$ into $\mathbf{l}^p$ for $p \geq 1$, i.e., $A \in V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^p)$, then A is a completely continuous operator from $\mathbf{l}^1$ into $\mathbf{l}^p$, $A \in V(\mathbf{l}^1 \rightarrow \mathbf{l}^p)$, and $V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^p) \subset V(\mathbf{l}^1 \rightarrow \mathbf{l}^p)$.

(a) A matrix operator A is continuous from $\mathbf{l}^1$ into $\mathbf{l}^p$, $A \in S(\mathbf{l}^1 \rightarrow \mathbf{l}^p)$, if and only if

$$\|A\| = \sup_{1 \leq j < \infty} \left(\sum_{i=1}^{\infty} |a_{ij}|^p \right)^{1/p} < \infty.$$

(b) A matrix operator A is ω -continuous from $\mathbf{l}^1$ into $\mathbf{l}^p$, $A \in V(\omega, \mathbf{l}^1 \rightarrow$

²⁶ A criterion for compactness of a matrix operator A acting in $\mathbf{l}^2$ has been introduced by Gilbert in [31],

$$\lim_{k \rightarrow \infty} \sup_{\sum_{j=1}^{\infty} x_j^2 \leq 1} \sup_{\sum_{i=1}^{\infty} y_i^2 \leq 1} \left| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} x_j y_i - \sum_{i=1}^k \sum_{j=1}^k a_{ij} x_j y_i \right| = 0,$$

and was generalized by Gribanov [30] for a class of ω -continuous matrix operators in arbitrary sequence spaces.

l^p), if and only if ²⁷

$$\|A\| = \lim_{k \rightarrow \infty} \sup_{k \leq j < \infty} \sum_{i=1}^{\infty} |a_{ij}|^p = 0.$$

The next theorem concerns properties of matrix operators in the space of bounded sequences l^∞ [32], [30].

Theorem 3.21. *If a matrix operator A is ω -continuous from l^∞ into l^∞ , $A \in V(\omega, l^\infty \rightarrow l^\infty)$, then A is a completely continuous operator from l^∞ into l^∞ , $A \in V(l^\infty \rightarrow l^\infty)$, and $V(\omega, l^\infty \rightarrow l^\infty) \subset V(l^\infty \rightarrow l^\infty)$.*

(a) *A matrix operator A is continuous from l^∞ into l^∞ , $A \in S(l^\infty \rightarrow l^\infty)$, if and only if*

$$\|A\| = \sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty.$$

(b) *A matrix operator A is ω -continuous from l^∞ into l^∞ , $A \in V(\omega, l^\infty \rightarrow l^\infty)$, if and only if*

$$\|A\| = \lim_{k \rightarrow \infty} \sup_{k \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}| = 0.$$

The next theorem gives criteria for matrix operators acting from l^1 into l^∞ .

Theorem 3.22. (a) *In order for a matrix operator A to be a continuous operator from l^1 into l^∞ , $A \in S(l^1 \rightarrow l^\infty)$, it is necessary and sufficient that*

$$\|A\| = \sup_{1 \leq i, j < \infty} |a_{ij}| < \infty.$$

²⁷A particular case for the compactness of $A : l^1 \rightarrow l^1$ is discussed in [17], such that

$$\|A\| = \lim_{k \rightarrow \infty} \sup_{k \leq j < \infty} \sum_{i=1}^{\infty} |a_{ij}| = 0. \quad (3.19)$$

Note that Griбанov's theorems for continuity and ω -continuity presented in this section, and the above criterion (3.19) for compactness of $A : l^1 \rightarrow l^1$, represent necessary and sufficient conditions for continuity and compactness of matrix operators acting in certain sequence spaces. Similar to the sufficient condition (strong condition) for compactness of $A : l^2 \rightarrow l^2$ (3.15), there is a sufficient condition for compactness of $A : l^1 \rightarrow l^1$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty. \quad (3.20)$$

This condition is much stronger than (3.19), and not every compact matrix operator $A : l^1 \rightarrow l^1$ satisfies condition (3.20).

(b) In order for a matrix operator A to be an ω -continuous operator from $\mathbf{l}^1$ into $\mathbf{l}^\infty$, $A \in V(\omega, \mathbf{l}^1 \rightarrow \mathbf{l}^\infty)$, it is necessary and sufficient that

$$\lim_{k \rightarrow \infty} \sup_{1 \leq j < \infty} \sup_{k \leq i < \infty} |a_{ij}| = 0,$$

$$\lim_{k \rightarrow \infty} \sup_{1 \leq i < \infty} \sup_{k \leq j < \infty} |a_{ij}| = 0.$$

Finally, there is a theorem for continuity and ω -continuity of matrix operators acting from $\mathbf{l}^p$ into $\mathbf{l}^\infty$, $p > 1$.

Theorem 3.23. *If a matrix operator A is ω -continuous from $\mathbf{l}^p$ into $\mathbf{l}^\infty$, $A \in V(\omega, \mathbf{l}^p \rightarrow \mathbf{l}^\infty)$, then A is a completely continuous operator from $\mathbf{l}^p$ into $\mathbf{l}^\infty$, $A \in V(\mathbf{l}^p \rightarrow \mathbf{l}^\infty)$, and $V(\omega, \mathbf{l}^p \rightarrow \mathbf{l}^\infty) \subset V(\mathbf{l}^p \rightarrow \mathbf{l}^\infty)$.*

(a) *A matrix operator A is continuous from $\mathbf{l}^p$ into $\mathbf{l}^\infty$, $A \in S(\mathbf{l}^p \rightarrow \mathbf{l}^\infty)$, if and only if*

$$\|A\| = \sup_{1 \leq i < \infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{1/q} < \infty,$$

where q is given by the relationship $1/p + 1/q = 1$.

(b) *A matrix operator A is an ω -continuous operator from $\mathbf{l}^p$ into $\mathbf{l}^\infty$, $A \in V(\omega, \mathbf{l}^p \rightarrow \mathbf{l}^\infty)$, if and only if*

$$\|A\| = \lim_{k \rightarrow \infty} \sup_{k \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}|^q = 0.$$

The properties of continuity and complete continuity (compactness) of matrix operators based on Gribanov's theorems, in connection with problems of uniqueness and solvability of infinite systems of linear equations, were extensively studied and reported in the Russian and Ukrainian literature; see, for example, [33]–[38].

3.8 Closed and Closable Operators

In this brief section we discuss, for completeness, the concept of closed and closable operators.

Definition 3.25. *The operator $A : H_1 \rightarrow H_2$ acting on a domain D_A is said to be closed if, for $\{x_n\} \subset D_A$, $x_n \rightarrow x$ and $y_n = Ax_n \rightarrow y$, then $x \in D_A$ and $y = Ax$.*

In some sense, the property of being closed is a weak form of continuity. For instance, let A be continuous and let $x_n \rightarrow x$, i.e., $\lim_{n \rightarrow \infty} x_n = x$. Then, $Ax = A \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Ax_n$, such that we can apply A term by term (we can interchange A and the limit operator), and the resulting

sequence will converge. Now, assume that A is merely closed and that $x_n \rightarrow x$. If A is applied term by term and the resulting sequence Ax_n converges to something (it need not), then $Ax = A \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Ax_n$.

To summarize, we can freely interchange continuous operators and limits. We can interchange closed operators and limits as long as the sequence obtained by the term-by-term action of the operator converges. It can be shown that adjoint operators are always closed, from which it follows that self-adjoint operators are always closed.

Alternatively, an operator $A : H_1 \rightarrow H_2$ is closed if its *graph*, defined to be the set of pairs

$$\Gamma(A) = \{(x, Ax) : x \in D_A\},$$

is a closed subspace of $H_1 \oplus H_2$.

If $\Gamma(A)$ is not closed, then it may have a closed extension (the requirement is that $\widetilde{\Gamma(A)}$ be the graph of an operator). In this case the operator is said to be *closable*. It can be shown that symmetric operators are closable. Furthermore, every continuous, i.e., bounded, linear operator is closed, but not every closed linear operator is continuous. For example, derivative operators are often closed, but are not bounded.

Some properties related to the concept of a closed operator are the following.

- A closed operator on a closed domain is bounded.
- A closed operator does not necessarily have a closed domain or range.
- The null space of a closed operator is closed.

3.9 Invertible Operators

3.9.1 General Concepts

The idea of an *operator inverse* arises naturally from the operator equation $Ax = b$, where one wants to solve for x given A and b . Division by an operator is not defined, but if A possesses a continuous inverse A^{-1} , then this equation has a unique solution, $x = A^{-1}b$, and this solution is a continuous function of b . Although in practice this is not the method by which one would numerically solve an equation of the form $Ax = b$, the concept of an inverse is a very important analytical tool in the theory of linear operators, both on finite-dimensional and infinite-dimensional spaces.

Definition 3.26. An operator $A : H_1 \rightarrow H_2$ is said to be invertible if there exists an operator $A^{-1} : H_2 \rightarrow H_1$ such that

$$A^{-1}Ax = x,$$

for all $x \in H_1$, and

$$AA^{-1}y = y$$

for all $y \in H_2$. The operator A^{-1} is called the inverse of A .

Note that $A^{-1}A = I_1$, the identity operator of H_1 , and $AA^{-1} = I_2$, the identity operator of H_2 . It is easy to see that if an inverse exists, it is unique.²⁸

We can also observe that if $A : H_1 \rightarrow H_2$ is invertible, with the inverse given by $A^{-1} : H_2 \rightarrow H_1$, then $A^{-1} : H_2 \rightarrow H_1$ is invertible, with the inverse being $A : H_1 \rightarrow H_2$.

Next we characterize some properties relating to invertible operators.

Definition 3.27. An operator $A : H_1 \rightarrow H_2$ is said to be one-to-one (injective) if no two distinct elements of H_1 are mapped to the same element of H_2 .

This states that $A : H_1 \rightarrow H_2$ is one-to-one if and only if $Ax_1 \neq Ax_2$ whenever $x_1 \neq x_2$ for $x_1, x_2 \in H_1$.

The next theorem presents an important statement concerning one-to-one linear operators.

Theorem 3.24. A linear operator $A : H_1 \rightarrow H_2$ is one-to-one if and only if $N_A = \{0\}$.

Proof. Assume A is one-to-one and let $x \in N_A$. Then $Ax = 0$, but $A0 = 0$, and since we assumed A is one-to-one, then $x = 0$ and so $N_A = \{0\}$. Now assume $N_A = \{0\}$. We need to show that if two points $x_1, x_2 \in H_1$ map into the same $y \in H_2$, then $x_1 = x_2$. Indeed, $Ax_1 = Ax_2$ implies $A(x_1 - x_2) = 0$, which in turn implies $(x_1 - x_2) \in N_A = \{0\}$. Therefore, $x_1 = x_2$. ■

If $N_A = \{0\}$ then A is said to have a *trivial null space*.

It seems obvious that if an operator is not one-to-one then it cannot be (uniquely) invertible. Indeed, one-to-oneness is a necessary condition for an operator to have an inverse. Necessary and sufficient conditions for A to be invertible are given next.

Theorem 3.25. An operator $A : H_1 \rightarrow H_2$ is invertible if and only if it is one-to-one and onto ($R_A = H_2$).

Such an operator, which is one-to-one (injective) and onto (surjective), is said to be *bijective*. From the above it can be seen that if $A : H_1 \rightarrow H_2$ is onto, then it is invertible if and only if $Ax = 0$ implies $x = 0$.

²⁸Let two inverses of an operator A be denoted by A_1^{-1} and A_2^{-1} , and y be any point of H_2 . Then $A_1^{-1}y = A_1^{-1}I_2y = A_1^{-1}AA_2^{-1}y = I_1A_2^{-1}y = A_2^{-1}y$.

For matrix operators we have the following useful criterion. A matrix operator $A : \mathbf{C}^n \rightarrow \mathbf{C}^n$ having entries $a_{ij} \in \mathbf{C}$ is said to be *strictly diagonally dominant* if

$$\sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| < |a_{ii}|$$

for $i = 1, 2, \dots, n$. In this case we have the following theorem.

Theorem 3.26. *If $A : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is strictly diagonally dominant, then $A^{-1} : \mathbf{C}^n \rightarrow \mathbf{C}^n$ exists.*

Proof. Let $A : \mathbf{C}^n \rightarrow \mathbf{C}^n$ be a strictly diagonally dominant matrix, and suppose that the inverse does not exist. Then, there exists $\mathbf{0} \neq \mathbf{x} \in \mathbf{C}^n$ such that $A\mathbf{x} = \mathbf{0}$. We therefore have $\sum_{j=1}^n a_{mj}x_j = 0$ for any $1 \leq m \leq n$. Let $x_m = \max_{1 \leq i \leq n} |x_i|$, and write $a_{mm}x_m = -\sum_{\substack{j=1 \\ j \neq m}}^n a_{mj}x_j$. Then, $|a_{mm}||x_m| = |\sum_{\substack{j=1 \\ j \neq m}}^n a_{mj}x_j| \leq |x_m| \sum_{\substack{j=1 \\ j \neq m}}^n |a_{mj}|$, which contradicts the assumption on diagonal dominance. ■

Basic Properties of Inverses

An important theorem in functional analysis is the following for bounded operators, which is typically cast in terms of Banach spaces.

Theorem 3.27. (*Open mapping theorem*) *A continuous bijection of one Banach space onto another Banach space has a continuous inverse.*

A merely one-to-one mapping (not necessarily onto) would perhaps seem to be invertible, yet the inverse can act on elements of H_2 that can not be “reached” from H_1 . One remedy of this problem is the restriction $D_{A^{-1}} = R_A$. Such an operator is intrinsically one-to-one and onto. In the following, when we state that the operator $A : H_1 \rightarrow H_2$ is invertible, we implicitly assume either that A is onto or that we are defining the domain of the inverse in a meaningful way. In this way the convenient conclusion is that *an operator A is invertible if and only if $Ax = 0$ implies $x = 0$* . Note though that unless $D_{A^{-1}} = R_A = H_2$, the open mapping theorem does not necessarily apply.

Theorem 3.28. *If $A : H_1 \rightarrow H_2$ is an invertible linear operator, then $A^{-1} : H_2 \rightarrow H_1$ is a linear operator.*

Proof. Let $y_1 = Ax_1$, $y_2 = Ax_2$, where $x_1, x_2 \in H_1$. Then $A^{-1}(y_1 + y_2) = A^{-1}(Ax_1 + Ax_2) = A^{-1}A(x_1 + x_2) = x_1 + x_2 = A^{-1}y_1 + A^{-1}y_2$. Similarly, $A^{-1}(\alpha y_1) = A^{-1}(\alpha Ax_1) = A^{-1}A(\alpha x_1) = \alpha x_1 = \alpha A^{-1}y_1$. ■

Theorem 3.29. *If $A : H_1 \rightarrow H_2$ is invertible and $\{x_1, x_2, \dots, x_n\} \subset H_1$ is a linearly independent set, then $\{Ax_1, Ax_2, \dots, Ax_n\} \subset H_2$ is a linearly independent set.*

Proof. Assume $\alpha_1 Ax_1 + \alpha_2 Ax_2 + \dots + \alpha_n Ax_n = 0$; then $A(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) = 0$. Because A is invertible, $(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) = 0$, which implies $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ since $\{x_n\}$ is a linearly independent set. Then we see that $\{Ax_1, Ax_2, \dots, Ax_n\}$ are linearly independent. ■

Theorem 3.30. *If $A : H_1 \rightarrow H_2$ and $B : H_2 \rightarrow H_3$ are invertible, then $BA : H_1 \rightarrow H_3$ is invertible with $(BA)^{-1} = A^{-1}B^{-1}$.*

Proof. If A and B are invertible, $(A^{-1}B^{-1})(BA) = A^{-1}(B^{-1}B)A = A^{-1}I_2A = A^{-1}A = I_1$. Then clearly BA is invertible $((BA)^{-1}(BA) = I_1$ and $(BA)(BA)^{-1} = I_3$) with $(BA)^{-1} = A^{-1}B^{-1}$. ■

Theorem 3.31. *If A is a positive definite invertible operator, A^{-1} is positive definite.*

Proof. Consider $Ax = y$ for all $y \in D_{A^{-1}}$ and some $x \in H$. Then $\langle A^{-1}y, y \rangle = \langle A^{-1}Ax, Ax \rangle = \langle x, Ax \rangle \geq 0$ (since A is positive definite). ■

Theorem 3.32. *If A is closed and A^{-1} exists, then A^{-1} is closed.*

Proof. Consider a sequence $\{y_n\} \subset R_A$ such that $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} A^{-1}y_n = x$. To prove that A^{-1} is closed we need to show that $y \in R_A$ and $A^{-1}y = x$. Since A is closed, $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} x_n = x$, then $x \in D_A$ and $Ax = y$. Therefore, $y \in R_A$ and $A^{-1}y = x$ (the inverse of A exists). ■

Theorem 3.33. *If A is closed and A^{-1} exists, then R_A is closed if and only if A^{-1} is bounded.*

Proof. First assume that A^{-1} is bounded and prove that R_A is closed. Consider a sequence $\{y_n\} \subset R_A$ such that $\lim_{n \rightarrow \infty} y_n = y$. The fact that A^{-1} is bounded shows that $x_n = A^{-1}y_n$ is a Cauchy sequence such that $\lim_{n \rightarrow \infty} x_n = x$ with $\{x_n\} \subset D_A$ and $\lim_{n \rightarrow \infty} Ax_n = y$. Since A is closed (according to the definition, $x \in D_A$ and $Ax = y$), this leads to the conclusion that $y \in R_A$ and therefore R_A is closed.

Now we assume that R_A is closed and prove that A^{-1} is bounded. The proof is based on the fact that a closed operator on a closed domain is bounded [12, p. 332]. Since A^{-1} is closed (from the above theorem) and R_A is closed, then A^{-1} is bounded. ■

Theorem 3.34. *Let $A : H_1 \rightarrow H_2$ be a bounded operator that is onto ($R_A = H_2$). If A has a bounded inverse, then A^* is invertible with $(A^*)^{-1} = (A^{-1})^*$.*

Proof. Comparing the following inner products for any $x \in H_1$ and $y \in H_2$,

$$\langle y, (A^{-1})^* A^* x \rangle = \langle A^{-1} y, A^* x \rangle = \langle A A^{-1} y, x \rangle = \langle y, x \rangle$$

and

$$\langle y, A^* (A^{-1})^* x \rangle = \langle A y, (A^{-1})^* x \rangle = \langle A^{-1} A y, x \rangle = \langle y, x \rangle,$$

we conclude that $(A^{-1})^* A^* x = A^* (A^{-1})^* x = x$; hence $(A^*)^{-1} = (A^{-1})^*$. ■

Theorem 3.35. *If A is a bounded self-adjoint operator, and A^{-1} is a bounded operator, then A^{-1} is self-adjoint.*

Proof. The proof simply follows from the previous theorem: $(A^{-1})^* = (A^*)^{-1} = A^{-1}$. ■

The next theorem is quite important, since it states necessary and sufficient conditions for an operator to have a bounded inverse.

Theorem 3.36. *A linear operator $A : H_1 \rightarrow H_2$ possesses a bounded inverse defined on R_A , $A^{-1} : R_A \rightarrow H_1$, if and only if there exists $m > 0$ such that $\|Ax\| \geq m \|x\|$ for all $x \in H_1$. Furthermore, $\|A^{-1}y\| \leq (1/m) \|y\|$.*

Such operators A are called bounded below operators (sometimes called *bounded away from zero*) (also see Section 3.5 for bounded below and above operators). Note that A itself does not need to be bounded.

Proof. Assume $\|Ax\| \geq m \|x\|$ is true and show A is invertible with a bounded inverse. If $Ax = 0$, then $\|0\| = 0 \geq m \|x\|$, so $\|x\| \leq 0$, but from the properties of a norm $\|x\| \geq 0$, so $\|x\| = 0$, which implies $x = 0$. Since the null space is trivial, then A is one-to-one, and since we are defining the domain of the inverse as the range of A (inherently onto), then A is invertible. To show the inverse is bounded, let $x = A^{-1}y$, which leads to $\|Ax\| = \|AA^{-1}y\| = \|y\| \geq m \|x\| = m \|A^{-1}y\|$, or $\|A^{-1}y\| \leq (1/m) \|y\|$, and so A^{-1} is bounded. Now assume A possesses a bounded inverse and show there exists an $m > 0$ such that $\|Ax\| \geq m \|x\|$. Indeed, if A^{-1} exists and is bounded, then $\|A^{-1}y\| = \|x\| \leq w \|y\| = w \|Ax\|$ or $\|Ax\| \geq (1/w) \|x\|$. ■

Since every linear operator on a finite-dimensional space is bounded, then if $A : H^n \rightarrow H^n$ is invertible, where H^n is finite-dimensional, then A^{-1} is bounded.

Well Posed Problems

Bounded below operators are important in defining what are known as *well posed problems*. The operator equation $Ax = b$ is called well posed if there exists a unique solution x that depends continuously on the “data” b . It is obvious what role existence and uniqueness play in a problem being well posed. The requirement that the solution be a continuous function of the data is important because one does not want small errors in the data to result in large errors in the solution.

If a problem is not well posed it is said to be *ill posed*. When A is bounded below, it possesses a continuous (bounded) inverse, such that $x = A^{-1}b$ depends continuously on b , and the problem is well posed (see also Section 3.11).

The next theorem shows that compact operators on infinite-dimensional spaces are not bounded below and hence do not have a continuous inverse.

Theorem 3.37. *If $A : H \rightarrow H$ for H an infinite-dimensional space is compact and invertible, then its inverse is unbounded.*

Proof. Let $\{x_n\}$ be an infinite orthonormal sequence in H . If $A : H \rightarrow H$ is compact, then $\lim_{n \rightarrow \infty} Ax_n = 0$ (see Section 3.6) and, therefore, the operator $A : H \rightarrow H$ is not bounded below. Based on Theorem 3.36, we conclude that the inverse operator $A^{-1} : H \rightarrow H$, where A is compact, is unbounded. ■

Therefore, while in the “forward direction” compact operators have a “nice” effect, in the “inverse direction” just the opposite is true. An analogy is that integration tends to smooth out functions, while the inverse operation, differentiation, tends to make functions less smooth. However, on a finite-dimensional space the inverse of a compact operator is bounded, by Theorem 3.5.

Examples of the above relevant to electromagnetics are first-kind compact integral operators with Green’s function kernels, where the inverse is a differential (unbounded) operator associated with a wave equation.

Resolvent Operators

An operator with very desirable properties in both the forward and inverse directions is the operator having the form $A - \lambda I$, where A is compact and $\lambda \neq 0$ (recall that I is not compact). In preparation for examining this type of operator, we state the following for $A - \lambda I$, with A not necessarily compact.

Theorem 3.38. *The operator $A - \lambda I$, with $A : H \rightarrow H$ bounded and $|\lambda| > \|A\|$, is invertible with bounded inverse. Furthermore,*

$$(A - \lambda I)^{-1} = - \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$$

and

$$\|(A - \lambda I)^{-1}\| \leq (|\lambda| - \|A\|)^{-1}.$$

The above is known as the *Neumann expansion* for $(A - \lambda I)^{-1}$. In general, the operator $(A - \lambda I)^{-1}$ is known as the *resolvent operator*. Before proving the above, it is convenient to prove the following.

Theorem 3.39. *If $A : H \rightarrow H$ is a bounded operator such that $\|A\| < 1$, then $(A - I)^{-1}$ exists and is bounded on H , and*

$$(A - I)^{-1} = -\sum_{n=0}^{\infty} A^n,$$

which is convergent in norm. Furthermore, $\|(A - I)^{-1}\| \leq (1 - \|A\|)^{-1}$.

Proof. The series $\sum_{n=0}^{\infty} \|A\|^n$ converges for $\|A\| < 1$ (geometric series, $\sum_{n=0}^{\infty} \|A\|^n = (1 - \|A\|)^{-1}$). Because $\|A^n\| \leq \|A\|^n$, then the series $\sum_{n=0}^{\infty} \|A^n\|$ converges. When a series of the form $\sum_n \|x\|$ converges, the series $\sum_n x$ is called *absolutely convergent*. As discussed on p. 133, the space of all bounded operators on H is itself a complete (Banach) space, and in a complete space absolute convergence implies convergence in norm. Thus, $\sum_{n=0}^{\infty} A^n$ converges in norm. To show that the inverse is given by $-\sum_{n=0}^{\infty} A^n$, assume this is the case and form $(A - I)(A - I)^{-1} = -(A - I)(I + A + A^2 + \cdots + A^n) = -(A^{n+1} - I)$. As $n \rightarrow \infty$, $A^{n+1} \rightarrow 0$ since $\|A\| < 1$, proving the relationship. Finally, we have $\|(A - I)^{-1}\| = \|\sum_{n=0}^{\infty} A^n\| \leq \sum_{n=0}^{\infty} \|A\|^n = (1 - \|A\|)^{-1}$. ■

The above theorem also holds for $A : X \rightarrow X$, with X being a Banach space. Now, by forming $(A - \lambda I)^{-1} = -(1/\lambda)(I - \frac{1}{\lambda}A)^{-1}$ we see from the last theorem that the series converges if $\|(1/\lambda)A\| < 1$, or $\|A\| < \lambda$, proving Theorem 3.38.

For A compact we have the following.

Theorem 3.40. *For $A : H \rightarrow H$ a compact operator and $\lambda \neq 0$, where λ is not an eigenvalue of A , then the operator $(A - \lambda I)^{-1}$ is bounded.*

One proof of this theorem follows from Theorem 4.13.

In particular, the case $\lambda = 1$ often arises in the theory of integral equations, where one finds that these so-called “identity plus compact” operators have very desirable properties.

Right and Left Inverses

Sometimes an operator is not invertible in the sense described above, but may possess what is known as a *right inverse* or a *left inverse*. These

concepts are not often utilized in applied electromagnetics, but a study of these operators helps to clarify the geometrical structure of invertible linear operators on Hilbert spaces.

Definition 3.28. An operator $A : H_1 \rightarrow H_2$ is said to be left invertible if an operator $A_L^{-1} : H_2 \rightarrow H_1$ exists such that

$$A_L^{-1}A = I_1,$$

where I_1 is the identity operator on H_1 . The operator $A_L^{-1} : H_2 \rightarrow H_1$ is said to be the left inverse of $A : H_1 \rightarrow H_2$.

Theorem 3.41. An operator $A : H_1 \rightarrow H_2$ is left invertible if and only if it is one-to-one.

Proof. First show that A is left invertible if it is one-to-one. Indeed, assuming A is one-to-one, then $A : H_1 \rightarrow R_A$ is invertible with $A^{-1} : R_A \rightarrow H_1$. Let $A_L^{-1} : H_2 \rightarrow H_1$ be any extension of A^{-1} . Then $A_L^{-1}A = I_1$. Now assume that A is left invertible and show it is one-to-one. Assume $Ax_1 = Ax_2$. Then $x_1 = A_L^{-1}(Ax_2) = A_L^{-1}Ax_2 = x_2$. ■

It can be seen from the proof that the left inverse is not necessarily unique, since the extension is not necessarily unique. If the operator A is onto, then A_L^{-1} is unique.

Definition 3.29. An operator $A : H_1 \rightarrow H_2$ is said to be right invertible if an operator $A_R^{-1} : H_2 \rightarrow H_1$ exists such that

$$AA_R^{-1} = I_2,$$

where I_2 is the identity operator on H_2 . The operator $A_R^{-1} : H_2 \rightarrow H_1$ is said to be the right inverse of $A : H_1 \rightarrow H_2$.

Theorem 3.42. An operator $A : H_1 \rightarrow H_2$ is right invertible if and only if it is onto ($R_A = H_2$).

Proof. It is easy to see that A is right invertible if it is onto. Now assume that A is right invertible and show it is onto. Assuming $y \in H_2$, then $y = AA_R^{-1}y = Ax$, so that y is in the range of A . Because $y \in H_1$, $y \in R_A$, and y is arbitrary, then $R_A = H_2$. ■

The right inverse is not necessarily unique, but if the operator is one-to-one then the right inverse is unique.

Theorem 3.43. If an operator $A : H_1 \rightarrow H_2$ is both right and left invertible, then the operator is invertible with $A^{-1} = A_L^{-1} = A_R^{-1}$.

Proof. The proof is trivial since if A has both a right and left inverse it is one-to-one and onto and hence invertible. ■

3.9.2 Examples Relating to Inverses

1. The identity operator ($Ix \equiv x$) on an infinite-dimensional space is invertible (bounded, yet not compact) with $I^{-1} = I$. The identity operator on a finite-dimensional space is bounded, compact, and invertible. The zero operator ($0x \equiv 0$) is not invertible.
2. The operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$ corresponding to the infinite matrix $[a_{ij}]$, where $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$, is bounded and compact. Then, if A is invertible, A^{-1} is unbounded. Note also that by Theorem 3.40, $(I - A)$ is invertible and the inverse operator $(I - A)^{-1}$ is bounded. This result is used in the solution of second-kind infinite systems of linear equations.
3. The operator $A : \mathbf{l}^2 \rightarrow \mathbf{l}^2$, defined by $A(\mathbf{x}) \equiv \{x_n/n^\alpha\}_{n=1}^{\infty}$, where $\alpha > 1/2$ and $\mathbf{x} = \{x_1, x_2, \dots\}$, is bounded since

$$\|A\mathbf{x}\| = \left(\sum_{n=1}^{\infty} \frac{|x_n|^2}{n^{2\alpha}} \right)^{1/2} \leq \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} = \|\mathbf{x}\|.$$

The operator A is also invertible: $A^{-1}(\mathbf{x}) = \{n^\alpha x_n\}_{n=1}^{\infty}$, but its inverse is not bounded. Indeed, considering the sequence $\{x_n\}$ of elements with $\|\mathbf{x}\| = 1$, it is clear that $\|A^{-1}(\mathbf{x})\| = n^\alpha \rightarrow \infty$ as $n \rightarrow \infty$.

4. The bounded operator $A : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is uniquely represented by the $n \times n$ matrix $[a_{ij}]$ where $a_{ij} = \langle Ax_j, x_i \rangle$, with $\{x_1, x_2, \dots, x_n\}$ a basis of $\mathbf{C}^n$. The operator is invertible (one-to-one and onto) if and only if $\det [a_{ij}] \neq 0$.
5. The bounded multiplication operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv f(t)x(t)$$

with $k = \max_{a \leq t \leq b} |f(t)| < \infty$ has as its inverse the division operator,

$$(A^{-1}x)(t) = \frac{1}{f(t)}x(t)$$

assuming $f(t) \neq 0$.

6. The differentiation operator $A : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$ defined by

$$(Ax)(t) \equiv \frac{d}{dt}x(t) = x'(t),$$

$$D_A \equiv \{x : x(t), x'(t) \in \mathbf{L}^2(a, b)\}$$

is not one-to-one (and so cannot be invertible), since there exist infinitely many functions, all differing from one another by a constant,

that have the same image. That is, $N_A = \{c\} \neq \{0\}$. However, if $a = -\infty$ and $b = \infty$, then constants are not in D_A and the operator is invertible. Also, for a, b finite, if we change the domain of A to be

$$D_A \equiv \{x : x(t), x'(t) \in \mathbf{L}^2(a, b), x(a) = 0\}.$$

then A is invertible, A^{-1} being of course the integration operator

$$(A^{-1}x)(t) = \int_a^t x(s) ds.$$

3.9.3 Green's Functions and Green's Operators

At this point it is worthwhile to draw special attention to operators inverse to differential operators, leading to the concept of a Green's function and a Green's operator. Green's functions for electromagnetics have been discussed in some detail in Chapter 1. They are also extensively discussed for an important class of second-order differential operators in Chapter 5 and in various sections in Part II. The discussion here is intended to establish the general concept of a Green's function as the kernel of an integral operator, the Green's operator, which is inverse to an invertible differential operator. The development is mostly formal and presented at a somewhat abstract level to avoid details that would draw attention from the main concepts.

Scalar Problems

Green's Operators

Consider a linear n th-order differential operator $L : \mathbf{L}^2(a, b) \rightarrow \mathbf{L}^2(a, b)$

$$-L \equiv p_n(t) \frac{d^n}{dt^n} + p_{n-1}(t) \frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t) \frac{d}{dt} + p_0(t),$$

$$D_L \equiv \left\{ \begin{array}{l} \psi : \frac{d^m \psi}{dt^m} \in \mathbf{L}^2(a, b), \quad m = 0, 1, 2, \dots, n, \\ B_1(\psi) = \eta_1, \quad B_2(\psi) = \eta_2 \end{array} \right\}$$

and the differential equation

$$(L - \lambda)\psi = \phi, \tag{3.21}$$

where λ is a complex parameter, $\psi = \psi(t) \in D_{L-\lambda} = D_L$, and $\phi = \phi(t) \in R_{L-\lambda}$. In the following we will also use the notation $L_\lambda \equiv L - \lambda$. Boundary

(and/or initial) conditions are

$$\begin{aligned}
 B_1(\psi) &= \alpha_{11}\psi(a) + \cdots + \alpha_{1n}\psi^{(n-1)}(a) + \beta_{11}\psi(b) + \cdots + \beta_{1n}\psi^{(n-1)}(b) \\
 &= \eta_1, \\
 &\vdots \\
 B_n(\psi) &= \alpha_{n1}\psi(a) + \cdots + \alpha_{nn}\psi^{(n-1)}(a) + \beta_{n1}\psi(b) + \cdots + \beta_{nn}\psi^{(n-1)}(b) \\
 &= \eta_n,
 \end{aligned}
 \tag{3.22}$$

with $\alpha_{i,j}, \beta_{i,j} \in \mathbf{R}$. We assume $L - \lambda$ is invertible (the example on p. 189 shows that not all differential operators are invertible). Formally solving (3.21) by applying the inverse operator leads to

$$\psi = (L - \lambda)^{-1} \phi.$$

Linearity of the differential operator implies linearity of $(L - \lambda)^{-1}$ by Theorem 3.28; therefore, one would expect that the solution ψ should be representable as a superposition integral²⁹

$$\psi(t) = (L - \lambda)^{-1} \phi(t) = \int_a^b g(t, t', \lambda) \phi(t') dt'. \tag{3.23}$$

This is also consistent with the fact that we would expect the inverse of a differential operator to be an integral operator. The function $g(t, t', \lambda)$ is called the *Green's function* of the operator $L - \lambda$. Often one considers the case where $\lambda = 0$, such that $g(t, t', 0) = g(t, t')$ is the Green's function.

If we substitute the solution (3.23) into (3.21), we obtain

$$\begin{aligned}
 \phi(t) &= (L - \lambda) \psi(t) = (L - \lambda) \int_a^b g(t, t', \lambda) \phi(t') dt' \\
 &= \int_a^b [(L - \lambda) g(t, t', \lambda)] \phi(t') dt',
 \end{aligned}$$

and we can make the identification

$$(L - \lambda) g(t, t', \lambda) = \delta(t - t'). \tag{3.24}$$

This will formally serve as the defining equation for the Green's function g in the sense of distributions. Also note that regardless of the possible inhomogeneity of the boundary conditions associated with L , the Green's function is taken to satisfy the homogeneous boundary conditions

$$B_1(g) = \cdots = B_n(g) = 0.$$

²⁹If $(a, b) = (-\infty, \infty)$, then the form of this solution is correct. In the presence of boundaries it may be necessary to add additional boundary contributions to the solution. These will occur when the boundary conditions that the Green's function and the quantity ψ obey differ from one another.

The inverse operator is called a *Green's operator*, denoted by G_λ , such that

$$\left((L - \lambda)^{-1} \phi\right)(t) = (G_\lambda \phi)(t) = \int_a^b g(t, t', \lambda) \phi(t') dt',$$

which has as its kernel the associated Green's function.

Adjoint Green's Operators

One can go through the same procedure with the adjoint equation

$$(L - \lambda)^* \psi^* = \phi^* \quad (3.25)$$

(note $(L - \lambda)^* = L^* - \bar{\lambda}$) where the domain of $L_\lambda^* \equiv (L - \lambda)^*$ is obtained from requiring that the bilinear concomitant in the generalized Green's theorem (3.3) vanish, leading to adjoint homogeneous boundary conditions

$$B_1^*(\psi^*) = \cdots = B_n^*(\psi^*) = 0$$

on ψ^* . The operator $(L - \lambda)^*$ satisfies

$$\langle (L - \lambda) \xi, \zeta \rangle = \langle \xi, (L - \lambda)^* \zeta \rangle$$

for $\xi \in D_L$, $\zeta \in D_{L^*}$.

The adjoint equation (3.25) leads to an *adjoint Green's operator*

$$\left(((L - \lambda)^*)^{-1} \psi\right)(t) = (G_\lambda^* \psi)(t) = \int_a^b g^*(t, t', \lambda) \psi(t') dt'$$

in terms of an *adjoint Green's function* $g^*(t, t', \lambda)$ that satisfies

$$(L - \lambda)^* g^*(t, t', \lambda) = \delta(t - t')$$

subject to adjoint homogeneous boundary conditions $B_1^*(g^*) = \cdots = B_n^*(g^*) = 0$. Forming $\langle L_\lambda \psi, \psi^* \rangle = \langle \psi, L_\lambda^* \psi^* \rangle$, it is seen that $\langle G_\lambda \psi, \phi \rangle = \langle \psi, G_\lambda^* \phi \rangle$, which leads to

$$\overline{g^*}(t, t', \lambda) = g(t', t, \lambda). \quad (3.26)$$

As an example, for the symmetric operator $L_\lambda = -d^2/dt^2 - \lambda$, we obtain the scalar free-space Green's functions

$$g(t, t') = \frac{e^{-i\sqrt{\lambda}|t-t'|}}{2i\sqrt{\lambda}},$$

$$g^*(t, t') = -\frac{e^{+i\sqrt{\lambda}|t-t'|}}{2i\sqrt{\lambda}}.$$

When L_λ is self-adjoint, then $g^* = g$ so that $\overline{g}(t, t', \lambda) = g(t', t, \lambda)$.

When L_λ is real and self-adjoint (a common case in electromagnetics), one can obtain a further, actually more useful, result, namely

$$g(t, t', \lambda) = g(t', t, \lambda). \quad (3.27)$$

To see this, note that if L_λ is self-adjoint then $L_\lambda^* g^*(t, t', \lambda) = L_\lambda g^*(t, t', \lambda) = \delta(t - t')$, and $\overline{L_\lambda g^*(t, t', \lambda)} = L_\lambda \overline{g^*}(t, t', \lambda) = \delta(t - t')$, since L_λ is real. From the last equality one sees that if the boundary conditions on g and those on $\overline{g^*}$ are identical, then $g(t, t', \lambda) = \overline{g^*}(t, t', \lambda)$, proving (3.27) from (3.26). The statement regarding the boundary conditions can be seen to hold since the boundary conditions on g and those on g^* will be identical by self-adjointness of L_λ and, assuming conditions of the form (3.22) on g^* with $\alpha_{i,j}, \beta_{i,j} \in \mathbf{R}$, $0 = B_i(g) = B_i(g^*) = \overline{B_i(g^*)} = B_i(\overline{g^*})$.

It is important to note that (3.27) also holds for L_λ real and formally self-adjoint (symmetric) if the boundary conditions on g and those on $\overline{g^*}$ are identical. For the operator leading to the free-space Green's function ($L = -\nabla^2$) this is the case, since the radiation condition for g (outgoing) is the conjugate of the radiation condition for g^* (incoming).³⁰ In electromagnetics one can also determine (3.27) based on reciprocity. If the medium is nonreciprocal, L_λ will not be self-adjoint and (3.27) will generally not hold.

It is worthwhile to point out that if L_λ is self-adjoint, one does not need to consider the adjoint problem; it is even more important to note that often the adjoint problem does not need to be considered even if L_λ is nonself-adjoint. Assuming L_λ is invertible, all one needs is an appropriate Green's theorem (which will be available whenever the operator is formally self-adjoint) applied to the elements ψ, g (see Section 1.3.4 and Chapter 5), whereby one obtains an explicit solution. The various Green's theorems are usually stated for L real and formally self-adjoint (e.g. $L = d^2/dt^2, \nabla^2, \nabla \times \nabla \times$); since this is the usual case in electromagnetics, it is usually not necessary to consider the adjoint problem.

However, if the homogeneous equation $L_\lambda \psi_0 = 0$ admits nonhomogeneous solutions, then $L_\lambda \psi = \phi$ cannot be solved uniquely, and perhaps not at all. To see this, note that if $L_\lambda \psi_0 = 0$ has nontrivial solutions, then by Theorem 3.25 L_λ is not invertible. Therefore, the Green's operator will not exist. However, in Section 3.11 we show that under these conditions $L_\lambda \psi = \phi$ will have a (nonunique) solution if and only if $\phi \perp \psi_0^*$, where ψ_0^* are solutions of the adjoint homogeneous equation, $L_\lambda^* \psi_0^* = 0$. In this case, what is known as an *extended* or *generalized* Green's operator can be found (see [39, p. 12] for details). Therefore, in these cases consideration of the adjoint problem will be necessary to ascertain solvability of the equations.

In summary, if L is formally self-adjoint (symmetric), then one need not consider the adjoint problem if L is invertible. Of course, if L is self-adjoint, the adjoint and original problems coincide. In more general cases one needs

³⁰However, to work in an L^2 space we consider the space to have small loss.

to consider the adjoint problem and utilize some form of generalized Green's theorem, e.g., (3.3).

As an illustration of obtaining an explicit solution, assume L_λ is real and formally self-adjoint (a common case in electromagnetics, e.g., $L = d^2/dt^2$). Then the generalized Green's theorem (3.4) is

$$\int_a^b (L_\lambda \psi) g dt = \int_a^b \psi (L_\lambda g) dt + J(\psi, g)|_a^b. \quad (3.28)$$

In terms of an inner product, (3.28) can be written as

$$\langle L_\lambda \psi, \bar{g} \rangle = \langle \psi, \overline{L_\lambda g} \rangle + J(\psi, g)|_a^b.$$

Using (3.21) and (3.24), and the sifting property of the delta-function, one obtains the solution as

$$\psi(t') = \int_a^b \phi(t) g(t, t', \lambda) dt - J(\psi, g)|_a^b. \quad (3.29)$$

It is important to note that any Green's function satisfying (3.24) will work in the above. The judicious choice of boundary conditions (always of a homogeneous form or satisfying some fitness condition) on g do affect the conjunct term, and often a preferable set of conditions is dictated by the goal of simplifying or eliminating the conjunct J .

Green's Operators as Improper Integral Operators

Although the procedure leading to (3.29) usually provides the correct solution in the scalar case, a few comments are required in generalizing the method to other problems of interest in electromagnetics.

The generalized Green's theorem will be valid for suitably defined elements, i.e., if $\psi, g, L_\lambda \psi$, and $L_\lambda g$ are suitably well behaved; in practice this means adequately differentiable. Regarding the discussion leading up to (1.52), one notes that the correct procedure that ensures the elements utilized in the Green's theorem are suitably well behaved is to apply the theorem to the intervals $[a, t' - \varepsilon)$ and $(t' + \varepsilon, b]$, thereby omitting the point $t = t'$ from the integration. If $\Omega = [a, b]$ and $\Omega_0 = (t' - \varepsilon, t' + \varepsilon)$, then (3.28) becomes

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_0} (L_\lambda \psi) g dt \\ = \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_0} \psi (L_\lambda g) dt + \lim_{\varepsilon \rightarrow 0} \left\{ J(\psi, g)|_a^{t' - \varepsilon} + J(\psi, g)|_{t' + \varepsilon}^b \right\}. \end{aligned}$$

In this case the first term on the right side vanishes (since $L_\lambda g = 0$ on $\Omega - \Omega_0$) and, replacing $L_\lambda \psi$ with ϕ , one obtains

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_0} \phi(t) g(t, t', \lambda) dt = \lim_{\varepsilon \rightarrow 0} \left\{ J(\psi, g)|_a^{t' - \varepsilon} + J(\psi, g)|_{t' + \varepsilon}^b \right\}.$$

By reorganizing the conjunct term as

$$J(\psi, g)|_a^{t'-\varepsilon} + J(\psi, g)|_{t'+\varepsilon}^b = J(\psi, g)|_a^b - J(\psi, g)|_{t'-\varepsilon}^{t'+\varepsilon},$$

we have

$$\lim_{\varepsilon \rightarrow 0} J(\psi, g)|_{t'-\varepsilon}^{t'+\varepsilon} = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_0} \phi(t) g(t, t', \lambda) dt + J(\psi, g)|_a^b, \quad (3.30)$$

where the limit on the conjunct term on the right side has been removed since ψ, g are assumed regular at the endpoints. By noting the similarity of (3.30) and (3.29), one expects

$$\lim_{\varepsilon \rightarrow 0} J(\psi, g)|_{t'-\varepsilon}^{t'+\varepsilon} = -\psi(t'), \quad (3.31)$$

leading to

$$\psi(t') = \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_0} \phi(t) g(t, t', \lambda) dt - J(\psi, g)|_a^b. \quad (3.32)$$

If the Green's function is weakly singular, then the solution (3.29) should be interpreted in the sense of (3.32).

The condition (3.31) characterizes the nature of the singularity of g [39, p. 10]. For a given problem one knows the conjunct and the properties of the Green's function, and (3.31) may be explicitly evaluated. For example, with $L = d^2/dt^2 = L^*$, (3.28) is simply Green's second theorem in one dimension, where the conjunct is $J(\psi, g) = g d\psi/dx - \psi dg/dx$. Explicitly, $(d^2/dt^2) g(t, t') = \delta(t - t')$ leads to $dg(t, t')/dt|_{t=t'-\varepsilon}^{t=t'+\varepsilon} = 1$ and continuity of g at $t = t'$, such that (3.31) is satisfied.

As described in Section 1.3.4, for most scalar and vector problems the limiting procedure is not necessary (in the sense that the correct answer nevertheless emerges from the sifting property of the delta-function), although the meaning of any resulting improper integral needs to be properly understood. In the dyadic case the limiting procedure, or some other rigorous method, needs to be used to obtain correct results.

Finally, for a general nonsymmetric problem we use (3.3) applied to ψ, g^* ,

$$\int_a^b (L_\lambda \psi) \overline{g^*} dt = \int_a^b \psi \overline{(L_\lambda^* g^*)} dt + J(\psi, \overline{g^*})|_a^b. \quad (3.33)$$

Formally substituting $L_\lambda \psi = \phi$, $L_\lambda^* g^* = \delta$ into (3.33) yields

$$\psi(t') = \int_{\Omega} \phi(t) \overline{g^*}(t, t') d\Omega - J(\psi, \overline{g^*})|_a^b$$

such that the solution is obtained in terms of the conjugate-adjoint Green's function. In this one-dimensional case, specific examples are shown in Section 5.3.

Systems of Dyadic Operators

Dyadic Green's Operators

Because electromagnetic theory is inherently a vector theory, one often needs to consider vector and dyadic differential operators and corresponding dyadic Green's operators with dyadic Green's function kernels.

Toward this end, we next consider a 2×2 matrix system of dyadic differential operators³¹ $\mathbf{L} : H \rightarrow H$ and the matrix differential operator equation³²

$$(\mathbf{L} - \lambda \mathbf{I}) \Psi = \Phi, \quad (3.34)$$

where

$$\mathbf{L} = \begin{bmatrix} \underline{\mathbf{L}}_{1,1} & \underline{\mathbf{L}}_{1,2} \\ \underline{\mathbf{L}}_{2,1} & \underline{\mathbf{L}}_{2,2} \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \underline{\mathbf{I}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{I}} \end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} \Psi_1(\mathbf{r}) \\ \Psi_2(\mathbf{r}) \end{bmatrix} \in D_{\mathbf{L}}, \quad \Phi = \begin{bmatrix} \Phi_1(\mathbf{r}) \\ \Phi_2(\mathbf{r}) \end{bmatrix} \in R_{\mathbf{L} - \lambda \mathbf{I}}$$

with Ψ_i, Φ_i vectors, $i = 1, 2$. Assuming $\mathbf{L}_\lambda \equiv (\mathbf{L} - \lambda \mathbf{I})$ is invertible, and repeating the procedure followed in the one-dimensional case, we identify $\mathbf{L}_\lambda^{-1} = \mathbf{G}_\lambda$ as the inverse (Green's) operator ($\mathbf{L}_\lambda^{-1} \mathbf{L}_\lambda \Psi = \Psi = \mathbf{L}_\lambda^{-1} \Phi$). The solution of (3.34) is obtained as

$$\begin{aligned} \Psi(\mathbf{r}) &= (\mathbf{G}_\lambda \Phi)(\mathbf{r}) = \int_{\Omega} \mathbf{g}(\mathbf{r}, \mathbf{r}', \lambda) \Phi(\mathbf{r}') d\Omega' \\ &= \int_{\Omega} \begin{bmatrix} \underline{\mathbf{g}}_{1,1}(\mathbf{r}, \mathbf{r}', \lambda) & \underline{\mathbf{g}}_{1,2}(\mathbf{r}, \mathbf{r}', \lambda) \\ \underline{\mathbf{g}}_{2,1}(\mathbf{r}, \mathbf{r}', \lambda) & \underline{\mathbf{g}}_{2,2}(\mathbf{r}, \mathbf{r}', \lambda) \end{bmatrix} \cdot \begin{bmatrix} \Phi_1(\mathbf{r}') \\ \Phi_2(\mathbf{r}') \end{bmatrix} d\Omega', \end{aligned} \quad (3.35)$$

where $\mathbf{G}_\lambda$ is a Green's operator with (matrix) Green's function kernel $\mathbf{g}(\mathbf{r}, \mathbf{r}', \lambda)$, the matrix entries of which are dyadic Green's functions. Explicitly using the scalar product notation, one can also write the above as

$$\begin{aligned} \Psi_1(\mathbf{r}) &= \int_{\Omega} \underline{\mathbf{g}}_{1,1}(\mathbf{r}, \mathbf{r}', \lambda) \cdot \Phi_1(\mathbf{r}') d\Omega' + \int_{\Omega} \underline{\mathbf{g}}_{1,2}(\mathbf{r}, \mathbf{r}', \lambda) \cdot \Phi_2(\mathbf{r}') d\Omega', \\ \Psi_2(\mathbf{r}) &= \int_{\Omega} \underline{\mathbf{g}}_{2,1}(\mathbf{r}, \mathbf{r}', \lambda) \cdot \Phi_1(\mathbf{r}') d\Omega' + \int_{\Omega} \underline{\mathbf{g}}_{2,2}(\mathbf{r}, \mathbf{r}', \lambda) \cdot \Phi_2(\mathbf{r}') d\Omega'. \end{aligned} \quad (3.36)$$

³¹The treatment here generally follows [58, Sec. 1.1]; see also [10]. The generalization to an $n \times n$ system is straightforward, as is the reduction to a single dyadic or vector equation.

³²By this notation we mean

$$\Theta \Lambda = \begin{bmatrix} \underline{\Theta}_{1,1} & \underline{\Theta}_{1,2} \\ \underline{\Theta}_{2,1} & \underline{\Theta}_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \underline{\Lambda}_1 \\ \underline{\Lambda}_2 \end{bmatrix} = \begin{bmatrix} \underline{\Theta}_{1,1} \cdot \underline{\Lambda}_1 + \underline{\Theta}_{1,2} \cdot \underline{\Lambda}_2 \\ \underline{\Theta}_{2,1} \cdot \underline{\Lambda}_1 + \underline{\Theta}_{2,2} \cdot \underline{\Lambda}_2 \end{bmatrix}$$

for elements of Θ, Λ being dyadics. For example, in electromagnetics the case $\underline{\Theta}_{i,j} = \alpha_{i,j} \nabla \times \underline{\mathbf{I}}$ often occurs. In the case of vector or scalar entries, appropriate multiplication rules apply.

As in the scalar case treated previously, if the region of interest Ω is finite then (3.36) may need to be augmented by boundary terms (the conjunct in a generalized Green's theorem), in this case boundary integrals, representing contributions from induced currents not accounted for by the Green's function. The presence or absence of these contributions will depend on the boundary conditions satisfied by the Green's function and the fields, and an explicit solution is obtained by utilizing an appropriate Green's theorem.

If we substitute the solution (3.36) into (3.34) we have

$$\begin{aligned}\Phi(\mathbf{r}) &= \mathbf{L}_\lambda \Psi(\mathbf{r}) = \mathbf{L}_\lambda \int_{\Omega} \mathbf{g}(\mathbf{r}, \mathbf{r}', \lambda) \Phi(\mathbf{r}') d\Omega' \\ &= \int_{\Omega} [\mathbf{L}_\lambda \mathbf{g}(\mathbf{r}, \mathbf{r}', \lambda)] \Phi(\mathbf{r}') d\Omega'\end{aligned}\quad (3.37)$$

and we can make the identification

$$\mathbf{L}_\lambda \mathbf{g}(\mathbf{r}, \mathbf{r}', \lambda) = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'). \quad (3.38)$$

For notational convenience we set $\lambda = 0$ such that the individual dyadic Green's functions are seen to satisfy

$$\begin{aligned}\underline{\mathbf{L}}_{1,1} \cdot \underline{\mathbf{g}}_{1,1} + \underline{\mathbf{L}}_{1,2} \cdot \underline{\mathbf{g}}_{2,1} &= \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'), \\ \underline{\mathbf{L}}_{1,1} \cdot \underline{\mathbf{g}}_{1,2} + \underline{\mathbf{L}}_{1,2} \cdot \underline{\mathbf{g}}_{2,2} &= \underline{\mathbf{0}}, \\ \underline{\mathbf{L}}_{2,1} \cdot \underline{\mathbf{g}}_{1,1} + \underline{\mathbf{L}}_{2,2} \cdot \underline{\mathbf{g}}_{2,1} &= \underline{\mathbf{0}}, \\ \underline{\mathbf{L}}_{2,1} \cdot \underline{\mathbf{g}}_{1,2} + \underline{\mathbf{L}}_{2,2} \cdot \underline{\mathbf{g}}_{2,2} &= \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'),\end{aligned}\quad (3.39)$$

which can be decoupled to yield

$$\begin{aligned}(\underline{\mathbf{L}}_{1,1} - \underline{\mathbf{L}}_{1,2} \cdot \underline{\mathbf{L}}_{2,2}^{-1} \cdot \underline{\mathbf{L}}_{2,1}) \cdot \underline{\mathbf{g}}_{1,1} &= \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'), \\ (\underline{\mathbf{L}}_{2,2} - \underline{\mathbf{L}}_{2,1} \cdot \underline{\mathbf{L}}_{1,1}^{-1} \cdot \underline{\mathbf{L}}_{1,2}) \cdot \underline{\mathbf{g}}_{2,2} &= \mathbf{I} \delta(\mathbf{r} - \mathbf{r}').\end{aligned}$$

assuming $\underline{\mathbf{L}}_{i,i}^{-1}$ are invertible. Note also that the original equations (3.34) can be decoupled as

$$\begin{aligned}(\underline{\mathbf{L}}_{1,1} - \underline{\mathbf{L}}_{1,2} \cdot \underline{\mathbf{L}}_{2,2}^{-1} \cdot \underline{\mathbf{L}}_{2,1}) \cdot \Psi_1 &= \Phi_1 - \underline{\mathbf{L}}_{1,2} \cdot \underline{\mathbf{L}}_{2,2}^{-1} \cdot \Phi_2, \\ (\underline{\mathbf{L}}_{2,2} - \underline{\mathbf{L}}_{2,1} \cdot \underline{\mathbf{L}}_{1,1}^{-1} \cdot \underline{\mathbf{L}}_{1,2}) \cdot \Psi_2 &= \Phi_2 - \underline{\mathbf{L}}_{2,1} \cdot \underline{\mathbf{L}}_{1,1}^{-1} \cdot \Phi_1.\end{aligned}$$

It can be seen that one advantage of the Green's function approach is that the source terms in the decoupled equations for the Green's functions are simpler than for the field quantities themselves.

Adjoint Dyadic Green's Operators

For the adjoint problem,

$$\mathbf{L}_\lambda^* \Psi^* = \Phi^* \quad (3.40)$$

and we identify $(\mathbf{L}_\lambda^*)^{-1} = \mathbf{G}_\lambda^*$ as the adjoint Green's operator with matrix kernel having entries $\underline{\mathbf{g}}_{\alpha,\beta}^*(\mathbf{r}, \mathbf{r}', \lambda)$, leading to the solution

$$\begin{aligned} \Psi^*(\mathbf{r}) &= (\mathbf{G}_\lambda^* \Phi^*)(\mathbf{r}) = \int_{\Omega} \mathbf{g}^*(\mathbf{r}, \mathbf{r}', \lambda) \Phi^*(\mathbf{r}') d\Omega' \\ &= \int_{\Omega} \begin{bmatrix} \underline{\mathbf{g}}_{1,1}^*(\mathbf{r}, \mathbf{r}', \lambda) & \underline{\mathbf{g}}_{1,2}^*(\mathbf{r}, \mathbf{r}', \lambda) \\ \underline{\mathbf{g}}_{2,1}^*(\mathbf{r}, \mathbf{r}', \lambda) & \underline{\mathbf{g}}_{2,2}^*(\mathbf{r}, \mathbf{r}', \lambda) \end{bmatrix} \cdot \begin{bmatrix} \Phi_1^*(\mathbf{r}') \\ \Phi_2^*(\mathbf{r}') \end{bmatrix} d\Omega'. \end{aligned} \quad (3.41)$$

where $\mathbf{g}^*$ satisfies

$$\mathbf{L}_\lambda^* \mathbf{g}^*(\mathbf{r}, \mathbf{r}', \lambda) = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \quad (3.42)$$

subject to adjoint boundary conditions. Also $\langle \mathbf{G}_\lambda \Psi, \Psi^* \rangle = \langle \Psi, \mathbf{G}_\lambda^* \Psi^* \rangle$, leading to

$$\underline{\mathbf{g}}_{\alpha,\beta}^*(\mathbf{r}, \mathbf{r}', \lambda) = \overline{\underline{\mathbf{g}}_{\beta,\alpha}^*(\mathbf{r}', \mathbf{r}, \lambda)}. \quad (3.43)$$

For example, the operator $L_\lambda = -\nabla^2 - \lambda$ leads to the free-space Green's function

$$g(\mathbf{r}, \mathbf{r}') = \frac{e^{-i\sqrt{\lambda}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

which satisfies an outgoing radiation condition (see Section 1.3.4), and the adjoint Green's function

$$g^*(\mathbf{r}, \mathbf{r}') = \frac{e^{+i\sqrt{\lambda}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

which satisfies an incoming radiation condition.

The preceding comments in the scalar case concerning the need to consider the adjoint problem hold for systems of vector or dyadic operators as well; if an appropriate Green's second theorem is available, one can express the solution in terms of the Green's function, without consideration of the adjoint problem, assuming $\mathbf{L}_\lambda$ is invertible. If $\mathbf{L}_\lambda$ is real and formally self-adjoint, then the generalized Green's theorem (3.4) is

$$\int_{\Omega} (\mathbf{L}_\lambda \Psi) \mathbf{g} d\Omega = \int_{\Omega} \Psi (\mathbf{L}_\lambda \mathbf{g}) d\Omega + \int_{\Gamma} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma. \quad (3.44)$$

Using (3.34) and (3.38) one obtains the solution as

$$\Psi(\mathbf{r}') = \int_{\Omega} \Phi(\mathbf{r}) \mathbf{g}(\mathbf{r}, \mathbf{r}', \lambda) d\Omega - \int_{\Gamma} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma. \quad (3.45)$$

As discussed in Section 1.3.5, the procedure leading to (3.45) may not result in the correct solution because of the strength of the dyadic Green's

function source-point singularity. A proper procedure is to apply (3.44) to $\Omega - \Omega_0$, where Ω_0 contains the point $\mathbf{r} = \mathbf{r}'$. We then have

$$\begin{aligned} \lim_{\Omega_0 \rightarrow 0} \int_{\Omega - \Omega_0} (\mathbf{L}_\lambda \Psi) \mathbf{g} d\Omega &= \lim_{\Omega_0 \rightarrow 0} \int_{\Omega - \Omega_0} \Psi (\mathbf{L}_\lambda \mathbf{g}) d\Omega \\ &\quad + \lim_{\Omega_0 \rightarrow 0} \int_{\Gamma} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma + \lim_{\Omega_0 \rightarrow 0} \int_{\Gamma_0} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma, \end{aligned} \quad (3.46)$$

where the first term on the right side vanishes since the point $\mathbf{r} = \mathbf{r}'$ is excluded from the integration. We obtain the solution as

$$\lim_{\Omega_0 \rightarrow 0} \int_{\Gamma_0} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma = \lim_{\Omega_0 \rightarrow 0} \int_{\Omega - \Omega_0} \Phi \mathbf{g} d\Omega - \int_{\Gamma} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma. \quad (3.47)$$

Comparing with (3.45) we identify

$$\lim_{\Omega_0 \rightarrow 0} \int_{\Gamma_0} \mathbf{J}(\Psi, \mathbf{g}) \cdot d\Gamma = \Psi(\mathbf{r}').$$

Examples

Consider Maxwell's differential equations for a region $\Omega \subset \mathbf{R}^3$ bounded by surface Γ (see, e.g., (1.13)),

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= -i\omega\mu\mathbf{H}(\mathbf{r}) - \mathbf{J}_m(\mathbf{r}), \\ \nabla \times \mathbf{H}(\mathbf{r}) &= i\omega\varepsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}_e(\mathbf{r}) \end{aligned}$$

subject to $\mathbf{n} \times \mathbf{E}|_{\Gamma} = \mathbf{0}$. One can consider the above in the form (3.34), where

$$\mathbf{L} = \begin{bmatrix} i\omega\varepsilon\mathbf{I} & -\nabla \times \mathbf{I} \\ \nabla \times \mathbf{I} & i\omega\mu\mathbf{I} \end{bmatrix}, \quad \Psi = \begin{bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{bmatrix}, \quad \Phi = \begin{bmatrix} -\mathbf{J}_e(\mathbf{r}) \\ -\mathbf{J}_m(\mathbf{r}) \end{bmatrix}$$

and $\lambda = 0$ (note the dyadic identity $(\nabla \times \mathbf{I}) \cdot \Psi = \nabla \times \Psi$). In this case we have

$$\mathbf{g}(\mathbf{r}, \mathbf{r}') = \begin{bmatrix} \underline{\mathbf{g}}_{e,e}(\mathbf{r}, \mathbf{r}') & \underline{\mathbf{g}}_{e,m}(\mathbf{r}, \mathbf{r}') \\ \underline{\mathbf{g}}_{m,e}(\mathbf{r}, \mathbf{r}') & \underline{\mathbf{g}}_{m,m}(\mathbf{r}, \mathbf{r}') \end{bmatrix},$$

and the formal solution (3.36) is

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= - \int_{\Omega} \underline{\mathbf{g}}_{e,e}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_e(\mathbf{r}') d\Omega' - \int_{\Omega} \underline{\mathbf{g}}_{e,m}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_m(\mathbf{r}') d\Omega', \\ \mathbf{H}(\mathbf{r}) &= - \int_{\Omega} \underline{\mathbf{g}}_{m,e}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_e(\mathbf{r}') d\Omega' - \int_{\Omega} \underline{\mathbf{g}}_{m,m}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_m(\mathbf{r}') d\Omega', \end{aligned} \quad (3.48)$$

where the Green's dyadics satisfy $\mathbf{Lg}(\mathbf{r}, \mathbf{r}') = \mathbf{I}\delta(\mathbf{r} - \mathbf{r}')$, leading to

$$\begin{aligned} -\nabla \times \underline{\mathbf{g}}_{m,e} + i\omega\varepsilon \underline{\mathbf{g}}_{e,e} &= \mathbf{I}\delta(\mathbf{r} - \mathbf{r}'), \\ -\nabla \times \underline{\mathbf{g}}_{m,m} + i\omega\varepsilon \underline{\mathbf{g}}_{e,m} &= \underline{\mathbf{0}}, \\ \nabla \times \underline{\mathbf{g}}_{e,e} + i\omega\mu \underline{\mathbf{g}}_{m,e} &= \underline{\mathbf{0}}, \\ \nabla \times \underline{\mathbf{g}}_{e,m} + i\omega\mu \underline{\mathbf{g}}_{m,m} &= \mathbf{I}\delta(\mathbf{r} - \mathbf{r}'), \end{aligned} \quad (3.49)$$

subject to $\mathbf{n} \times \underline{\mathbf{g}}_{e,e}|_{\Gamma} = \underline{\mathbf{0}}$ and $\mathbf{n} \times \underline{\mathbf{g}}_{e,m}|_{\Gamma} = \underline{\mathbf{0}}$.

Decoupling the above leads to

$$\begin{aligned} \nabla \times \nabla \times \underline{\mathbf{g}}_{e,e} - \omega^2 \mu \varepsilon \underline{\mathbf{g}}_{e,e} &= i\omega\mu \mathbf{I}\delta(\mathbf{r} - \mathbf{r}'), \\ \nabla \times \nabla \times \underline{\mathbf{g}}_{m,m} - \omega^2 \mu \varepsilon \underline{\mathbf{g}}_{m,m} &= i\omega\varepsilon \mathbf{I}\delta(\mathbf{r} - \mathbf{r}'), \end{aligned} \quad (3.50)$$

where $\mathbf{n} \times \underline{\mathbf{g}}_{e,e}|_{\Gamma} = \underline{\mathbf{0}}$ and $\mathbf{n} \times \nabla \times \underline{\mathbf{g}}_{m,m}|_{\Gamma} = \underline{\mathbf{0}}$, with $\underline{\mathbf{g}}_{e,m}$ and $\underline{\mathbf{g}}_{m,e}$ subsequently determined from the solution of (3.50) using the relationships (3.49). Alternatively, wave equations

$$\begin{aligned} \nabla \times \nabla \times \underline{\mathbf{g}}_{e,m} - \omega^2 \mu \varepsilon \underline{\mathbf{g}}_{e,m} &= \nabla \times \mathbf{I}\delta(\mathbf{r} - \mathbf{r}'), \\ \nabla \times \nabla \times \underline{\mathbf{g}}_{m,e} - \omega^2 \mu \varepsilon \underline{\mathbf{g}}_{m,e} &= -\nabla \times \mathbf{I}\delta(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (3.51)$$

subject to $\mathbf{n} \times \underline{\mathbf{g}}_{e,m}|_{\Gamma} = \underline{\mathbf{0}}$ and $\mathbf{n} \times \nabla \times \underline{\mathbf{g}}_{m,e}|_{\Gamma} = \underline{\mathbf{0}}$ can be solved, and the remaining quantities determined from (3.49). It can also be seen from the defining equations that the field quantities satisfy

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) &= -i\omega\mu \mathbf{J}_e(\mathbf{r}) - \nabla \times \mathbf{J}_m(\mathbf{r}), \\ \nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k^2 \mathbf{H}(\mathbf{r}) &= -i\omega\varepsilon \mathbf{J}_m(\mathbf{r}) + \nabla \times \mathbf{J}_e(\mathbf{r}) \end{aligned}$$

subject to $\mathbf{n} \times \mathbf{E}|_{\Gamma} = \mathbf{0}$, $\mathbf{n} \times \nabla \times \mathbf{H}|_{\Gamma} = \mathbf{0}$.

Without detailing the adjoint problem, it can be seen that the symmetry relation (3.43) reduces to

$$\underline{\mathbf{g}}_{\alpha,\beta}(\mathbf{r}, \mathbf{r}') = (-1)^{\alpha+\beta} \underline{\mathbf{g}}_{\beta,\alpha}^{\top}(\mathbf{r}', \mathbf{r}) \quad (3.52)$$

since $\nabla \times \nabla \times$ subject to boundary conditions of the form discussed is a real, self-adjoint operator. If radiation conditions, rather than boundary conditions, are specified, the operator is real and symmetric, and the same conclusion holds per the discussion subsequent to (3.27). The sign change is obvious from the form of the defining equations.

Finally, consider the scalar (transmission line) system (see Chapter 7)

$$\begin{aligned} \frac{dv(z)}{dz} &= -i\omega L i(z) - i_m(z), \\ \frac{di(z)}{dz} &= -i\omega C v(z) + i_e(z), \end{aligned} \quad (3.53)$$

which can be analyzed as a special case of the above. We have (3.34) with $\lambda = 0$,

$$\mathbf{L} = \begin{bmatrix} -i\omega C & -\frac{d}{dz} \\ \frac{d}{dz} & i\omega L \end{bmatrix}, \quad \mathbf{\Psi} = \begin{bmatrix} v(z) \\ i(z) \end{bmatrix}, \quad \mathbf{\Phi} = \begin{bmatrix} -i_e(z) \\ -i_m(z) \end{bmatrix},$$

leading to

$$\mathbf{g}(z, z') = \begin{bmatrix} g_{v,v}(z, z') & g_{v,i}(z, z') \\ g_{i,v}(z, z') & g_{i,i}(z, z') \end{bmatrix}$$

with the solution of (3.36) as

$$\begin{aligned} v(z) &= - \int_a^b g_{v,v}(z, z') i_e(z) dz' - \int_a^b g_{v,i}(z, z') i_m(z) dz', \\ i(z) &= - \int_a^b g_{i,v}(z, z') i_e(z) dz' - \int_a^b g_{i,i}(z, z') i_m(z) dz'. \end{aligned} \quad (3.54)$$

The Green's functions are seen to satisfy

$$\mathbf{Lg}(z, z') = \mathbf{I}\delta(z - z'),$$

where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the usual identity matrix, leading to

$$\begin{aligned} -\frac{d}{dz}g_{i,v} - i\omega Cg_{v,v} &= \delta(z - z'), \\ -\frac{d}{dz}g_{i,i} - i\omega Cg_{v,i} &= 0, \\ \frac{d}{dz}g_{v,v} + i\omega Lg_{i,v} &= 0, \\ \frac{d}{dz}g_{v,i} + i\omega Lg_{i,i} &= \delta(z - z'). \end{aligned}$$

Decoupling the above leads to

$$\begin{aligned} \left(\frac{d^2}{dz^2} + \omega^2 LC \right) g_{v,v}(z, z') &= -i\omega L\delta(z - z'), \\ \left(\frac{d^2}{dz^2} + \omega^2 LC \right) g_{i,i}(z, z') &= -i\omega C\delta(z - z'), \end{aligned}$$

where the voltage and current quantities satisfy

$$\begin{aligned} \left(\frac{d^2}{dz^2} + \omega^2 LC \right) v(z) &= -i\omega Li_e - i'_m, \\ \left(\frac{d^2}{dz^2} + \omega^2 LC \right) i(z) &= +i\omega Ci_m + i'_e. \end{aligned}$$

Note that

$$g_{\alpha,\beta}(z, z') = (-1)^{\alpha+\beta} g_{\beta,\alpha}(z', z).$$

3.10 Projection Operators

We have previously introduced the projection theorem (p. 120), which provides the idea of uniquely representing an element $z \in H$ as $z = x_0 + y_0$, where $x_0 \in M$, with M a closed linear subspace of H , and $y_0 \in M^\perp$. After introducing the concept of a basis for an infinite-dimensional system, it was stated that $x_0 = \sum_n \langle z, x_n \rangle x_n$ and $y_0 = \sum_n \langle z, y_n \rangle y_n$, where $\{x_n\}$ is an orthonormal basis for M and $\{y_n\}$ is an orthonormal basis for $M^\perp$. We can now cast this into operator form using *projection operators*.

Definition 3.30. Let M be a closed linear subspace of a Hilbert space H , and let $z = x_0 + y_0$, where $z \in H$, $x_0 \in M$, and $y_0 \in M^\perp$. An operator P defined on H is said to be the orthogonal projection onto M if

$$P(z) = P(x_0 + y_0) = x_0.$$

Projection operators are linear, bounded operators with³³ $\|P\| \leq 1$. In addition, $R_P = M$ and $N_P = M^\perp$ (note that $N_P \perp R_P$ and $H = R_P \oplus N_P$). Also, $I - P$ on H is an orthogonal projection onto $M^\perp$, i.e.,

$$(I - P)(z) = z - x_0 = y_0$$

with $R_{I-P} = M^\perp$ and $N_{I-P} = M$.

Some properties of projection operators follow.

- Projection operators are *idempotent*.³⁴
- A bounded operator is a projection if and only if it is idempotent and self-adjoint.
- A projection operator is a nonnegative operator.
- The sum $P + Q$ of two projection operators P and Q is a projection if and only if $PQ = 0$.
- The product PQ of two projection operators P and Q is a projection if and only if³⁵ $[P, Q] = 0$.
- A projection onto M , a closed linear subspace of H , is compact if and only if M is finite-dimensional.
- If P is a projection onto M , a closed linear subspace of H , then for all $x \in H$, $\langle Px, x \rangle = \|Px\|^2$.
- N_P and R_P are closed linear subspaces of H .

From the projection theorem one can see that the orthogonal projection P onto M is unique.

³³The Pythagorean formula gives $\|Pz\|^2 = \|x_0\|^2 = \|z - y_0\|^2 = \|z\|^2 - \|y_0\|^2 \leq \|z\|^2$.

³⁴An operator A is idempotent if $A^2 = A$.

³⁵Recall that $[P, Q] = PQ - QP$ is the commutator of operators P and Q .

Examples of Projection Operators

Consider an element $z \in H$ and a closed linear subspace M of H with $\{x_n\}$ an orthonormal basis for M . The projection operator onto M is defined as $Pz = \sum_n \langle z, x_n \rangle x_n$.

1. The zero operator is a projection operator onto $\{0\}$ ($M = \{0\}$, $M^\perp = H$).
2. The identity operator is a projection operator onto H ($M = H$, $M^\perp = \{0\}$),

$$Iz = z = \sum_n \langle z, x_n \rangle x_n.$$

Obviously, $\|I\| = 1$, and, in fact, $\|P\| = 1$ for every nonzero projection operator since $Px = x$ for all $x \in M$.

3. On $H = \mathbf{R}^3$ with the usual xyz -coordinate system, let M be the xy -plane with its standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. The orthogonal projection of $\gamma = (3\mathbf{e}_1, 2\mathbf{e}_2, 5\mathbf{e}_3)$ onto M is $P\gamma = \sum_{n=1}^2 \langle \gamma, \mathbf{e}_n \rangle \mathbf{e}_n = (3\mathbf{e}_1, 2\mathbf{e}_2)$.
4. On $H = \mathbf{L}^2(-\pi, \pi)$ let M^e be the closed linear subspace consisting of all even functions in H , defined by $x(t) = x(-t)$ for all t . Then $(M^e)^\perp = M^o$, the space of all odd functions in H (i.e., $x(t) = -x(-t)$ for all t). Then, letting $x(t) \in H$,

$$P^e x = \frac{1}{2} [x(t) + x(-t)]$$

is an orthogonal projection onto M^e , and

$$P^o x = (I - P^e) x = \frac{1}{2} [x(t) - x(-t)]$$

is an orthogonal projection onto M^o .

5. On $H = \mathbf{L}^2(-\pi, \pi)$ with basis

$$\left\{ \sqrt{1/(2\pi)}, \sqrt{1/\pi} \cos(nt), \sqrt{1/\pi} \sin(nt), n = 1, 2, \dots \right\}$$

let

$$\begin{aligned} x_0^e &= \sqrt{\frac{1}{2\pi}}, & x_n^e &= \sqrt{\frac{1}{\pi}} \cos(nt), \\ x_0^o &= 0, & x_n^o &= \sqrt{\frac{1}{\pi}} \sin(nt), \end{aligned}$$

$n = 1, 2, \dots$. Let M^e be the closed linear subspace consisting of all even functions in H , for which $\{x_n^e\}$ is a basis. Then $(M^e)^\perp = M^o$, the space of all odd functions in H , for which $\{x_n^o\}$ is a basis ($\{x_n^e\} \cup \{x_n^o\}$ is a basis for H). Then, letting $x(t) \in H$,

$$P^e x = \sum_{n=0}^{\infty} \langle x, x_n^e \rangle x_n^e$$

is the projection operator onto M^e , and

$$\begin{aligned} P^o x &= (I - P^e) x = \sum_{n=0}^{\infty} \langle x, x_n^e + x_n^o \rangle x_n - \sum_{n=0}^{\infty} \langle x, x_n^e \rangle x_n^e \\ &= \sum_{n=0}^{\infty} \langle x, x_n^o \rangle x_n^o \end{aligned}$$

is the projection operator onto M^o . Since the projection is unique,

$$P^e x = \frac{1}{2} [x(t) + x(-t)] = \sum_{n=0}^{\infty} \langle x, x_n^e \rangle x_n^e,$$

and similarly for P^o .

Projection operators are extensively used to show geometrical properties of spectral problems. In fact, advanced concepts in spectral theory are often cast in terms of projections. Therefore, much of the next chapter could be studied using projection operators, although for the purposes of studying spectral problems in applied electromagnetics these concepts will not be pursued much further. The interested reader can consult [23, Ch. 6], [56], [7]. For applications of projection operators in numerical solutions of electromagnetic operator equations, see [57], [43].

3.11 Solution of Operator Equations

In this section we consider the solution of linear equations at an operator level. In general, we are interested in operator equations of the form

$$(A - \lambda I) x = y, \quad (3.55)$$

where $A : H \rightarrow H$ is a linear operator, $x \in D_{A-\lambda I} = D_A \subseteq H$, and $y \in R_{A-\lambda I} \subseteq H$. If $\lambda \neq 0$, we call (3.55) an operator equation of the *second kind*, whereas if $\lambda = 0$, we have an operator equation of the *first kind*. Obviously, if the resolvent operator exists, formally we have the solution $x = (A - \lambda I)^{-1} y$, which is continuously dependent on y if the resolvent is bounded. Furthermore, if λ is an eigenvalue of A , we cannot solve (3.55) for $y \neq 0$, since the homogeneous form defines the eigenvalue problem, $(A - \lambda_n I) x_n = 0$.

3.11.1 Existence of Solutions

In general, it is difficult to state solvability theorems for (3.55) unless considerable restrictions are placed on A . In the following we present a solvability theorem known as the *Fredholm alternative*, which we state for the

operator equation $Ax = y$ (this is not necessarily a first-kind equation since A may have the form $A = B - \lambda I$). Operators satisfying the Fredholm alternative are called *Fredholm operators*. The alternative will be stated for operators in Hilbert spaces although a similar alternative holds for operators in Banach spaces.

Of course, for the equation $Ax = y$ to make sense we must have $x \in D_A$ and $y \in R_A$. In particular, the requirement $y \in R_A$ is important if $R_A \neq H$. Note also that if A is positive definite, the solution of $Ax = y$ can be shown to be equivalent to minimization of a quadratic functional [4, Ch. 5], although this topic will not be discussed here.

Theorem 3.44. (*Fredholm alternative*) *Let $A : H \rightarrow H$ be a bounded linear Fredholm operator. Then either*

I. for every $y, g \in H$ the nonhomogeneous equations

$$\begin{aligned} Ax &= y, \\ A^*f &= g \end{aligned} \tag{3.56}$$

have unique solutions x, f , and the corresponding homogeneous equations

$$\begin{aligned} Ax &= 0, \\ A^*f &= 0 \end{aligned} \tag{3.57}$$

have only the trivial solutions $x = 0$ and $f = 0$, or

II. the homogeneous equations

$$\begin{aligned} Ax &= 0, \\ A^*f &= 0 \end{aligned} \tag{3.58}$$

have the same number of (nontrivial) linearly independent solutions, $x_1, x_2, \dots, x_n$ and $f_1, f_2, \dots, f_n$, respectively, $n \geq 1$, and the nonhomogeneous equations

$$\begin{aligned} Ax &= y, \\ A^*f &= g \end{aligned}$$

are not solvable for all y, g . A solution (nonunique) exists if and only if y and g are such that $y \perp N_{A^}$ and $g \perp N_A$, respectively.*

The first part of the alternative is easily understood from Theorem 3.25 and its subsequent discussion, because if the homogeneous equations (3.57) have only trivial solutions, then A^{-1} and $(A^*)^{-1}$ exist, and so (3.56) can be uniquely solved as $x = A^{-1}y$ and $f = (A^*)^{-1}g$. The fact that both the original equation and the adjoint equation behave in the same manner, in terms of being solvable or not, can be seen to be reasonable, at least for the special case of the Fredholm operator $I - K$ with K compact (discussed next), from Theorem 3.17.

The second part of the alternative is less obvious, because in the case of nontrivial homogeneous solutions to $Ax = 0$, solvability of $Ax = y$ depends, perhaps unexpectedly, on properties of the adjoint operator A^* . The necessity of the condition $y \perp N_{A^*}$ for $Ax = y$ to be solvable in this case is seen from the following. Consider $Ax = y$ and $A^*f = 0$, and form the inner product $\langle Ax, f \rangle = \langle y, f \rangle$. Using $\langle y, f \rangle = \langle Ax, f \rangle = \langle x, A^*f \rangle = \langle x, 0 \rangle = 0$ we see that we must have the condition $\langle y, f \rangle = 0$, i.e., $y \perp N_{A^*}$. The sufficiency of this condition for Fredholm operators can also be proved, but will be omitted here. Of course, if A is self-adjoint, the Fredholm alternative is simplified, and no aspect of the adjoint operator need be considered.

For Theorem 3.44 to be useful, we need to know what types of operators satisfy the Fredholm alternative. Three classes of Fredholm operators are presented below.

We first state that the operator $A - \lambda I$, where $\lambda \neq 0$ and A is a compact operator, is a Fredholm operator, such that solvability of the equation $(A - \lambda I)x = y$ can be examined using the above alternative.

Theorem 3.45. *Let $A : H \rightarrow H$ be a compact linear operator and let $A_\lambda \equiv A - \lambda I$, where $\lambda \neq 0$. Then A_λ satisfies the Fredholm alternative.*

The proof is shown in [40, p. 451] in a Banach-space setting and in [12, p. 359] in a Hilbert space. An important case is provided by $\lambda = 1$. Alternatively, since $\lambda \neq 0$, one can consider the operator $(1/\lambda)A - I$ or $K - I$, where $(1/\lambda)A = K$ is compact if A is compact. See also Theorem 4.34.

Theorem 3.45 is most often applied to examine solvability of second-kind integral equations involving compact integral operators. The above theorem can be extended in the sense that A_λ satisfies the Fredholm alternative if $(A_\lambda)^n$, $n \geq 1$, is compact [41, p. 138]. Also note that $\dim N_A = \dim N_{A^*} < \infty$.

From the above it is easily seen that for A compact and $\lambda \neq 0$, if $(A - \lambda I)x = 0$ has only the trivial solution $x = 0$ (meaning λ is not an eigenvalue of A), then $(A - \lambda I)x = y$ has a unique solution given by $x = (A - \lambda I)^{-1}y$. From Theorem 3.40 one can see that the solution depends continuously on y , and so the problem is well posed.

The next theorem characterizes when the operator (matrix) equation $Ax = b$ is solvable, where $A : H^n \rightarrow H^m$ is bounded and compact (necessarily so since H^n and H^m are finite-dimensional Hilbert spaces) with $x \in H^n$ and $b \in H^m$.

Theorem 3.46. *The operator $A : H^n \rightarrow H^m$ satisfies the Fredholm alternative.*

Usually this is stated for $A : \mathbf{C}^n \rightarrow \mathbf{C}^m$, leading to the matrix system $Ax = b$ where A is an $m \times n$ complex matrix and b is an $m \times 1$ matrix. The

most familiar case is of course when $m = n$, such that $Ax = b$ is uniquely solvable if and only if $\det A \neq 0$, implying that $Ax = 0$ has only the trivial solution.

Lastly, in the case of a differential operator we have the following [12, p. 229].

Theorem 3.47. *The self-adjoint Sturm–Liouville differential operator L (see (5.1), (5.2), and (5.4)) satisfies the Fredholm alternative.*

Note also that the nonself-adjoint Sturm–Liouville operator with homogeneous boundary conditions, considered in Section 5.3, also satisfies the Fredholm alternative. The solution of operator equations is further discussed in Section 4.4 from the viewpoint of eigenfunction expansions.

3.11.2 Convergence of Solutions

While it is not the intent of this text to present a through discussion of numerical solution techniques, it is worthwhile to briefly list a few important facts concerning the numerical solution of operator equations (see, for instance, [14], [42], and [43]).

Recall from Sections 2.5.6 and 2.5.7 that given a Hilbert space H with an orthonormal basis $\{x_n\}$, any $x \in H$ may be written as a norm-convergent series $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$. This can be viewed as defining an N -term approximation $x \simeq x^N$,

$$x^N = \sum_{n=1}^N \langle x, x_n \rangle x_n, \quad (3.59)$$

where x^N lies in the N -dimensional subspace of H spanned by $\{x_n\}_{n=1}^N$. The error in the approximation is defined as $e^N \equiv x - x^N$, and because $\{x_n\}$ is a basis for H , $\lim_{N \rightarrow \infty} \|e^N\| = \lim_{N \rightarrow \infty} \|x - x^N\| = 0$. Furthermore, the error is orthogonal to the approximation, i.e.,

$$\langle x - x^N, x^N \rangle = 0. \quad (3.60)$$

For finite N , the coefficient $\langle x, x_n \rangle$ in (3.59) provides the minimum error in the approximation of x (i.e., this coefficient provides the best possible approximation to x in the N -dimensional subspace spanned by $\{x_n\}_{n=1}^N$).

Now, consider determining the unknown x from the general operator equation

$$Ax = y,$$

where $A : H_1 \rightarrow H_2$ with $\{x_n\}$ and $\{y_n\}$ bases for the spaces H_1 and H_2 ,

respectively. We form the approximations of $x \in H_1$ and $y \in H_2$ as³⁶

$$\begin{aligned} x^N &= \sum_{n=1}^N \alpha_n x_n, \\ y^N &= \sum_{n=1}^N \langle y, y_n \rangle y_n, \end{aligned} \quad (3.61)$$

where α_n are unknown coefficients to be determined, and we are assured that $y^N \rightarrow y \in H_2$ in norm. The general idea is to determine the solution of

$$Ax^N = y^N$$

for a given N , and consider under what conditions this solution converges to the solution of the original equation (i.e., when does $x^N \rightarrow x$).

Since the summation is finite, $Ax^N = \sum_{n=1}^N \alpha_n Ax_n$. Noting that $Ax_n \in H_2$, we have $Ax_n = \sum_{m=1}^{\infty} \langle Ax_n, y_m \rangle y_m \simeq (Ax_n)^N = \sum_{m=1}^N \langle Ax_n, y_m \rangle y_m$, so that $Ax^N = y^N$ becomes

$$\sum_{n=1}^N \alpha_n \sum_{m=1}^N \langle Ax_n, y_m \rangle y_m = \sum_{n=1}^N \langle y, y_n \rangle y_n,$$

leading to the matrix system

$$\sum_{n=1}^N \alpha_n \langle Ax_n, y_m \rangle = \langle y, y_m \rangle \quad (3.62)$$

for $m = 1, 2, \dots, N$. The solution of (3.62) provides the coefficients α_n , leading to determination of x^N as in (3.61).

For a given, finite-value N , the coefficients α_n obtained from (3.62) do not necessarily provide the best approximation of x in H_1 . That is, if we knew the exact solution x by other means, we would find that the coefficients α_n do not generally agree with the coefficients $\langle x, x_n \rangle$ in (3.59). Therefore, the approximation of the solution x is not the best possible approximation in H_1 ; however, we do obtain the best possible approximation of $Ax \in H_2$. Moreover, if we increase N , we do not necessarily know that the coefficients α_n converge to $\langle x, x_n \rangle$, although they may do so.

Alternatively, if we take

$$\begin{aligned} x &= \sum_{n=1}^{\infty} \alpha_n x_n, \\ y &= \sum_{n=1}^{\infty} \langle y, y_n \rangle y_n, \end{aligned} \quad (3.63)$$

³⁶We take the upper summation limit as N in both cases, which will lead to an $N \times N$ matrix system. However, this is not necessary, and one may generate an $N \times M$ matrix system solvable by least-squares or comparable methods.

then if A is bounded (to permit the interchange of A and the summation in (3.63)), we directly obtain the infinite matrix system

$$\sum_{n=1}^{\infty} \alpha_n \langle Ax_n, y_m \rangle = \langle y, y_m \rangle \quad (3.64)$$

for $m = 1, 2, \dots$. In theory, this infinite system yields the correct coefficients $\alpha_n = \langle x, x_n \rangle$ to make the equality in (3.63) valid. However, in order to solve (3.64) in practice, this system is truncated to size $N \times N$, leading to (3.62); either way we do not have assurance that the coefficients α_n converge to $\langle x, x_n \rangle$.

Method of Moments

In electromagnetics, the matrix equation (3.62) is usually obtained by a procedure known as the *method of moments* [44]. In this case we form an approximation

$$x \simeq x^N = \sum_{n=1}^N \alpha_n x_n,$$

where $\{x_n\} \in D_A$ are linearly independent functions called *expansion functions*. We therefore have $Ax \simeq \sum_{n=1}^N \alpha_n Ax_n$, and, rather than expanding Ax_n in the basis of H_2 as above, one forms a matrix equation by forcing the *residual* (error) $\sum_{n=1}^N \alpha_n Ax_n - y$ to be orthogonal to a set of *weighting* (or *testing*) functions $\{y_n\} \subset R_A$, i.e.,

$$\left\langle \sum_{n=1}^N \alpha_n Ax_n - y, y_m \right\rangle = 0$$

for $m = 1, 2, \dots, N$. This is equivalent to

$$\sum_{n=1}^N \alpha_n \langle Ax_n, y_m \rangle = \langle y, y_m \rangle, \quad (3.65)$$

which is the same as (3.62). If the expansion and testing functions are identical, then this is known as *Galerkin's method*. In practice a pseudo inner product is often used in (3.65).

The interpretation of the method-of-moments procedure is somewhat different from the former method, because we don't necessarily require $\{x_n\}$ or $\{y_n\}$ to be bases, or even orthogonal. However, as with the former method, we are finding the best approximation of $Ax \in R_A$ for a given set $\{y_n\}$ and in this sense minimizing the norm of the residual error. Therefore, to achieve a reasonably accurate solution of the operator equation in an engineering sense, we may not necessarily require the expansion functions to be complete in D_A , but they should be such that Ax_n is complete in R_A [45].

Convergence of Solutions

Unfortunately, in applications it is often not possible to show in a rigorous manner convergence of the approximation $x^N \rightarrow x$. However, two important and practical cases can be distinguished which lead to guaranteed convergence. We first consider the special case of a positive definite operator [46], [4].

Theorem 3.48. *Consider the operator equation $Ax = y$, where $A : H \rightarrow H$ is a positive definite operator. If $\{x_n\} = \{y_n\}$ forms an orthonormal basis for H_e , the Hilbert space associated with the energy norm (3.13), then*

$$x^N = \sum_{n=1}^N \alpha_n x_n \rightarrow x$$

in the norm of H , where α_n are determined by the solution of (3.65).

Proof. Since A is positive definite, we rewrite (3.65) in terms of the energy inner product (3.12) as

$$\sum_{n=1}^N \alpha_n \langle x_n, x_m \rangle_e = \langle x, x_m \rangle_e \quad (3.66)$$

for $m = 1, 2, \dots, N$. Because $\{x_n\}$ is an orthonormal set in H_e , then $\alpha_n = \langle x, x_n \rangle_e$ and we have $x^N = \sum_{n=1}^N \langle x, x_n \rangle_e x_n$. Using completeness of $\{x_n\}$ in H_e leads to³⁷ $\lim_{N \rightarrow \infty} \|x^N - x\|_e = 0$. By Theorem 3.15, since A is positive definite, convergence in energy leads to strong (norm) convergence in H , i.e., $\lim_{N \rightarrow \infty} \|x^N - x\| = 0$. ■

In electromagnetics, it often occurs that static problems lead to positive definite operators [47], such as the Laplacian operator considered in Chapter 6.

If the operator equation in question is a second-kind Fredholm equation, then convergence can also be established [14, Ch. 5], [42], [43, Ch. 5] (see also Theorem 3.18).

Theorem 3.49. *Consider the second-kind operator equation $(\lambda I - K)x = y$, where $K : H \rightarrow H$ is a compact operator and λ is not an eigenvalue of K . If $\{x_n\}$ and $\{y_n\}$ are orthonormal bases for H , then*

$$x^N = \sum_{n=1}^N \alpha_n x_n \rightarrow x$$

in norm, where α_n are determined by the solution of (3.65), with $A = (\lambda I - K)$.

³⁷If A is merely positive, then we must stop here and settle for weak convergence (by Theorem 3.15, part b). See also Theorem 3.14, which concerns forming a positive operator from any bounded operator that possesses an adjoint.

In fact, the above statement is more restrictive than it need be, and certain relaxations on the expansion and testing functions may be allowed. Thus, second-kind compact integral equations that admit solutions may often be solved by discretization, where convergence of the solution can be established. This important fact is the impetus of many regularization schemes.

Under the conditions of Theorem 3.49, (3.65) is simply a truncated version of (3.64). To examine convergence at the matrix level, consider the infinite second-kind matrix equation

$$x_i - \sum_{j=1}^{\infty} k_{ij} x_j = y_i, \quad (3.67)$$

$i = 1, 2, \dots$, where the matrix $[k_{ij}]$ is compact. We write (3.67) as

$$X - KX = Y$$

where $X = [x_1, x_2, \dots]^T$, $Y = [y_1, y_2, \dots]^T$, and $K = [k_{ij}]_{i,j=1}^{\infty}$. We also consider the "truncated" matrix system

$$X^n - K^n X^n = Y$$

where K^n is filled with zeroes outside of the $n \times n$ square. Then,

$$\begin{aligned} (X - KX) - (X^n - K^n X^n) &= 0, \\ X - X^n &= KX - K^n X^n \\ &= KX - K^n X^n + K^n X - K^n X, \\ (X - X^n) + K^n (X^n - X) &= (K - K^n) X. \end{aligned}$$

Therefore

$$(X - X^n) = (I - K^n)^{-1} (K - K^n) X,$$

and, upon taking norms,

$$\begin{aligned} \|(X - X^n)\| &= \|(I - K^n)^{-1} (K - K^n) X\| \\ &\leq \|(I - K^n)^{-1}\| \|K - K^n\| \|X\|, \end{aligned}$$

leading to

$$\frac{\|(X - X^n)\|}{\|X\|} \leq \|(I - K^n)^{-1}\| \|K - K^n\|$$

which can provide an error estimate for X^n . Furthermore, it can be shown that³⁸ $\|(I - K^n)^{-1} - (I - K)^{-1}\| \rightarrow 0$, and so as $n \rightarrow \infty$ the first

³⁸If $A : H \rightarrow H$ is bounded and invertible, $B : H \rightarrow H$ is bounded, and $\|A - B\| < \frac{1}{\|A^{-1}\|}$, then [17, p. 71] B is invertible and

$$\|A^{-1} - B^{-1}\| \leq \frac{\|A^{-1}\|^2 \|A - B\|}{1 - \|A^{-1}\| \|A - B\|}.$$

term on the right side becomes $\|(I - K)^{-1}\|$, which is bounded by Theorem 3.40 (this is where compactness enters into the problem). Obviously $\|K - K^n\| \rightarrow 0$ and so $X^n \rightarrow X$ in norm. The proof can also be done at the function space level if one can approximate the operator K by a sequence of finite-rank operators K^n such that $\|K - K^n\| \rightarrow 0$.

Replacing A with $(I - K)$ and B with $(I - K^n)$ proves the statement.